Calculation of the Ionospheric Potential in Steady-State and Non-Steady-State Models of the Global Electric Circuit

Nikolay N. Slyunyaev[1]*, Alexey V. Kalinin[1, 2], Evgeny A. Mareev[1] and Artem A. Zhidkov[1, 2]

[1]Institute of Applied Physics RAS, Nizhny Novgorod, Russia
[2]Lobachevsky State University of Nizhny Novgorod, Nizhny Novgorod, Russia

ABSTRACT: Spherical stationary and quasi-stationary models of the global electric circuit (GEC) are developed and analysed. Some subtle issues concerning the well-posedness of the corresponding initial and boundary value problems are discussed. It is shown that the ionospheric potential can always be uniquely determined from the solution, whereas if it is specified explicitly, the corresponding problem is ill-posed. Steady-state spherical models of the GEC are compared with plane-parallel models, and it is shown that the former have a number of advantages over the latter; in particular, it is demonstrated that the ionospheric potential cannot be determined within the 'flat-Earth' approximation, unless additional assumptions are made. Several exact and approximate analytical formulae for the ionospheric potential are derived according to steady-state spherical, non-steady-state spherical and steady-state plane-parallel models of the GEC. The connection between spherical models and classical multi-column models of the GEC is established.

INTRODUCTION

One of the most important experimental evidences of the global electric circuit (GEC) hypothesis is the observation that at every moment of time the potential difference between the lower ionosphere and the Earth's surface measured at remote locations is of the same magnitude. This result makes it possible to introduce the ionospheric potential, the potential of the outer boundary of the atmosphere (which can thus be considered equipotential) relative to the Earth's surface, which is one of the most fundamental characteristics of the GEC. Its significance is reinforced by the fact that it can be directly measured through aircraft and balloon soundings [e.g., *Markson*, 2007], which makes it one of the few well-studied characteristics of the global circuit, despite the measurements made until now being unsystematic and incomplete.

The long-term variation of the ionospheric potential corresponds to the dynamics of conductivity inhomogeneities determined by various natural and anthropogenic factors, and that is why the ionospheric potential can be used as an indicator of different physical processes in the atmosphere.

In this paper we compare the main approaches to modelling the GEC, analyse them from the perspective of well-posedness and place particular emphasis on the calculation of the ionospheric potential, for which several explicit formulae are derived according to different approaches.

THE MAIN EQUATONS AND THE MAIN ASSUMPTIONS

In order to analyse plane-parallel and spherical models of the GEC and to shed light on the crucial difference between them, let us start with Maxwell's equations. Whatever the model geometry is, we suppose that the atmosphere is bounded by the Earth's surface Σ_1 and a surface Σ_2 representing the lower limit of the ionosphere. We also suppose that both the dielectric permittivity and the magnetic permeability of the

*Corresponding author, email: slyunyaev.n@gmail.com, Postal address: Institute of Applied Physics RAS, 46 Ulyanova St., 603950 Nizhny Novgorod, Russia.

atmosphere are equal to 1, in which case non-stationary Maxwell's equations read as follows†:

$$\operatorname{curl}\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J}, \tag{1}$$

$$\operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \tag{2}$$

$$\operatorname{div}\mathbf{H} = 0, \tag{3}$$

$$\operatorname{div}\mathbf{E} = 4\pi\rho, \tag{4}$$

where $\mathbf{E}(\mathbf{r},\, t)$ is the electric field, $\mathbf{H}(\mathbf{r},\, t)$ is the magnetic field, $\mathbf{J}(\mathbf{r},\, t)$ is the current density, $\rho(\mathbf{r},\, t)$ is the charge density, $\mathbf{r}$ denotes the spatial coordinates, t stands for the time and c stands for the speed of light. These equations must be supplemented with Ohm's law

$$\mathbf{J} = \sigma\mathbf{E} + \mathbf{J}^{\mathrm{ext}}, \tag{5}$$

where $\sigma(\mathbf{r},\, t)$ is the conductivity and $\mathbf{J}^{\mathrm{ext}}(\mathbf{r},\, t)$ represents the external current density, as well as with the initial and boundary conditions. An important circumstance is that we regard thunderstorms as distributed current sources and suppose that the external current density is non-zero only within thunderclouds and other electrified clouds [e.g., *Volland*, 1984]. In other words, at every moment of time the positions of thunderstorms correspond to the spatial distribution of $\mathbf{J}^{\mathrm{ext}}$.

Assuming that both surfaces Σ_1 and Σ_2 are perfect conductors, we obtain the boundary conditions

$$E_\tau|_{\Sigma_1} = 0, \qquad E_\tau|_{\Sigma_2} = 0, \tag{6}$$

where the index τ indicates the tangential component. The initial condition may be written in the form

$$\mathbf{E}|_{t=0} = \mathbf{E}^0, \tag{7}$$

$\mathbf{E}^0(\mathbf{r})$ being a stationary electric field satisfying (6).

In this paper we analyse this problem in the quasi-stationary approximation—that is to say, we neglect the variation of $\mathbf{H}$ with time in (2), which yields the equation

$$\operatorname{curl}\mathbf{E} = 0. \tag{8}$$

Substituting the relation (5) into (1) gives

$$\operatorname{curl}\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\big(\sigma\mathbf{E} + \mathbf{J}^{\mathrm{ext}}\big). \tag{9}$$

Given σ and $\mathbf{J}^{\mathrm{ext}}$, the equations (6)–(9) form a system of equations in $\mathbf{E}$ and $\operatorname{curl}\mathbf{H}$. If $\operatorname{curl}\mathbf{H}$ and $\mathbf{E}$ are found, one can obtain the magnetic field itself and the space charge density from (3) and (4), provided that necessary boundary conditions for $\mathbf{H}$ are established. Therefore it is sufficient to find $\mathbf{E}$ for solving the entire system of equations (1)–(7) within the quasi-stationary approximation.

What we want now is to eliminate the magnetic field from the equation (9). This is a subtle issue, for it is here that plane-parallel models of the GEC differ from spherical models. Within the 'flat-Earth' approach the boundary surfaces Σ_1 and Σ_2 are supposed to be unbounded and the atmosphere is represented by an infinite slab between them, whereas in the spherical geometry both boundary surfaces are bounded, Σ_1 being encompassed by Σ_2, and the atmosphere is a thick shell confined by them (see Fig. 1). This topological distinction determines the difference in the model equations.

†Hereafter we use the Gaussian unit system.

For brevity, let us denote the right-hand side of (9) by $\mathbf{X}$. It can be shown that for the plane-parallel geometry the following two statements are equivalent:

(i) There exists a vector field $\mathbf{H}$ such that $\mathbf{X} = \operatorname{curl} \mathbf{H}$.

(ii) $\operatorname{div} \mathbf{X}$ is equal to zero.

However, for the spherical geometry the equivalent of (i) is not (ii) but

(iii) $\operatorname{div} \mathbf{X}$ is equal to zero, as is the flux of $\mathbf{X}$ through Σ_1.

(see, e.g., *Girault and Raviart* [1986] for a more detailed discussion of this issue). From this general observation we obtain that the equation (9) is equivalent to the equation

$$\frac{\partial\,(\operatorname{div} \mathbf{E})}{\partial t} + 4\pi \operatorname{div}(\sigma \mathbf{E}) = -4\pi \operatorname{div} \mathbf{J}^{\text{ext}} \tag{10}$$

for plane-parallel models and to the pair of equations (10) and

$$\oint_{\Sigma_1} \left(\frac{\partial \mathbf{E}}{\partial t} + 4\pi\sigma\mathbf{E} \right) d\mathbf{S} = -4\pi \oint_{\Sigma_1} \mathbf{J}^{\text{ext}}\, d\mathbf{S} \tag{11}$$

for spherical models. This means that, on the one hand, any field $\mathbf{E}$ satisfying (9) must also satisfy (10) (or (10) and (11)—depending on the geometry chosen), and, on the other hand, for any field $\mathbf{E}$ satisfying (10) (or (10) and (11)) there must exist a field $\mathbf{H}$ such that (9) holds. Therefore once we find all solutions to (10) (or (10) and (11)), we immediately find all solutions to (9), but, since the original Maxwell's equations also require that $\mathbf{H}$ satisfy (3) and meet the (unspecified here) boundary conditions, it is possible that although a solution to (10) (or (10) and (11)) satisfies (9) with some $\mathbf{H}$, yet it does not correspond to any solution of Maxwell's equations. However, we can replace this $\mathbf{H}$ by $\mathbf{H} + \operatorname{grad} \chi$ and try to satisfy all the necessary conditions by the appropriate choice of χ. It is easy to verify that we will arrive at Poisson's equation for χ with certain boundary conditions, and we can expect it to have a solution. Nevertheless, as we do not want to specify the boundary conditions for $\mathbf{H}$, in this paper we will not go into such an analysis and will only use the aforementioned fact that any solution to the original Maxwell's equations must also satisfy (10) (or (10) and (11)).

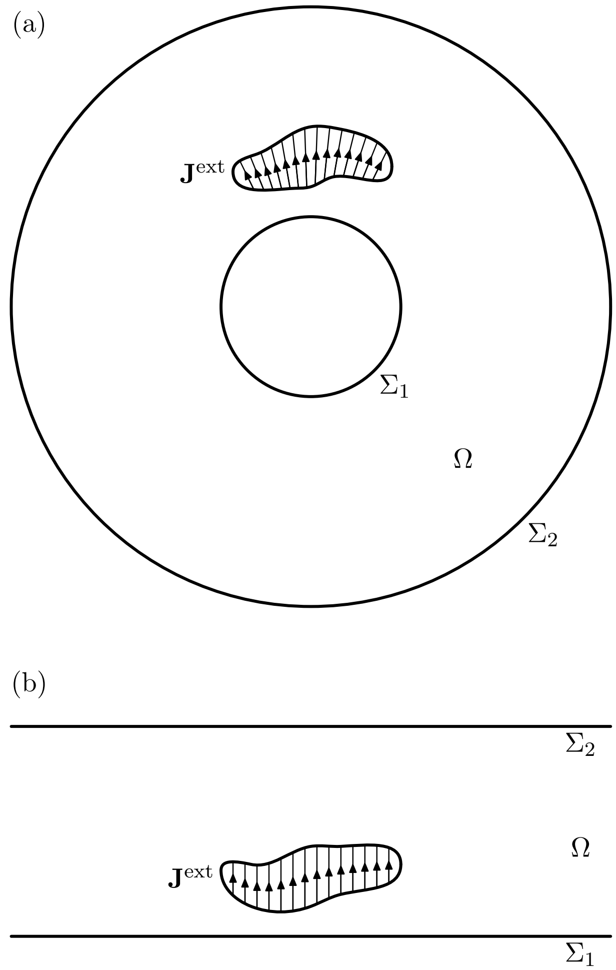

Figure 1: Geometry of spherical (a) and plane-parallel (b) models.

For both the plane-parallel and spherical model geometries the equation (8) makes it possible to introduce the electric potential, the function $\phi(\mathbf{r},\, t)$ such that $\mathbf{E} = -\operatorname{grad} \phi$, and to reformulate other equations in terms of this function. The conditions (6) mean that ϕ does not vary over each of the two boundary surfaces, and thus if it is set equal to zero at Σ_1, its value at Σ_2 represents the ionospheric potential V_{i}. The condition (7) requires that at the moment $t = 0$ ϕ be equal to $\phi^0(\mathbf{r})$, the potential of $\mathbf{E}^0$ satisfying

$\phi^0\big|_{\Sigma_1} = 0$. For spherical models of the GEC, combining these relations with the equations (10) and (11), we arrive at the system of equations

$$\frac{\partial}{\partial t}\Delta\phi + 4\pi\,\mathrm{div}\,(\sigma\,\mathrm{grad}\,\phi) = 4\pi\,\mathrm{div}\,\mathbf{J}^{\mathrm{ext}}, \tag{12}$$

$$\oint_{\Sigma_1}\left(\frac{\partial}{\partial t}\,\mathrm{grad}\,\phi + 4\pi\sigma\,\mathrm{grad}\,\phi\right)d\mathbf{S} = 4\pi\oint_{\Sigma_1}\mathbf{J}^{\mathrm{ext}}\,d\mathbf{S}, \tag{13}$$

$$\phi|_{\Sigma_1} = 0, \qquad \phi|_{\Sigma_2} = V_{\mathrm{i}}, \tag{14}$$

$$\phi|_{t=0} = \phi^0, \tag{15}$$

where $V_{\mathrm{i}}(t)$ is a certain function of the time. A similar system for the corresponding steady-state problem may be written as follows:

$$\mathrm{div}\,(\sigma\,\mathrm{grad}\,\phi) = \mathrm{div}\,\mathbf{J}^{\mathrm{ext}}, \tag{16}$$

$$\oint_{\Sigma_1}(\sigma\,\mathrm{grad}\,\phi)\,d\mathbf{S} = \oint_{\Sigma_1}\mathbf{J}^{\mathrm{ext}}\,d\mathbf{S}, \tag{17}$$

$$\phi|_{\Sigma_1} = 0, \qquad \phi|_{\Sigma_2} = V_{\mathrm{i}}, \tag{18}$$

where V_{i} is a certain constant. For plane-parallel models of the GEC the equations are the same, except for (13) and (17) which should be left out.

In some spherical models of atmospheric electricity the integral conditions (13) and (17) are omitted, the ionospheric potential is specified explicitly and the potential distribution is found by solving the problem (12), (14), (15) or (16), (18). However, such an approach is incorrect, inasmuch as the conditions (13) and (17) are direct consequences of Maxwell's equations, and it can be shown that once they are established, the ionospheric potential can always be uniquely determined from the equations (12)–(15) or (16)–(18). A detailed explanation of this issue is given in the next section.

SPHERICAL MODELS OF THE GEC

General remarks and well-posedness

Because of the spherical geometry of the atmosphere, the most natural approach to modelling the global circuit is to solve the electric field equations in the shell between the Earth's surface Σ_1 and the lower boundary of the ionosphere Σ_2. In the non-stationary case the electric field potential ϕ satisfies the equations (12)–(15), and in the stationary case, the equations (16)–(18). What is the most important is that the ionospheric potential V_{i} in (14) and (18) is not an independent parameter or function (in the non-stationary case V_{i} depends on t) but can always be determined from the solution [*Kalinin et al.*, 2014]. Let us demonstrate this for the non-stationary case (similar argument works for the stationary case).

Suppose there are two solutions to (12)–(15) ($\sigma(\mathbf{r},\,t)$, $\mathbf{J}^{\mathrm{ext}}(\mathbf{r},\,t)$ and $\phi^0(\mathbf{r})$ are assumed to be the same in both cases), $\phi^{(1)}(\mathbf{r},\,t)$ with the ionospheric potential $V_{\mathrm{i}}^{(1)}(t)$ and $\phi^{(2)}(\mathbf{r},\,t)$ with the ionospheric potential $V_{\mathrm{i}}^{(2)}(t)$. Setting $\delta\phi = \phi^{(1)} - \phi^{(2)}$ and subtracting the equations for $\phi^{(2)}$ from their counterparts for $\phi^{(1)}$, we arrive at the equations

$$\frac{\partial}{\partial t}\Delta\delta\phi + 4\pi\,\mathrm{div}\,(\sigma\,\mathrm{grad}\,\delta\phi) = 0,$$

$$\oint_{\Sigma_1}\left(\frac{\partial}{\partial t}\,\mathrm{grad}\,\delta\phi + 4\pi\sigma\,\mathrm{grad}\,\delta\phi\right)d\mathbf{S} = 0,$$

$$\delta\phi|_{\Sigma_1} = 0, \qquad \delta\phi|_{\Sigma_2} = \delta V_{\mathrm{i}},$$

$$\delta\phi|_{t=0} = 0,$$

where $\delta V_{\mathrm{i}}(t) = V_{\mathrm{i}}^{(1)}(t) - V_{\mathrm{i}}^{(2)}(t)$. The region occupied by the atmosphere being denoted by Ω, for any two moments of time t_1 and t_2 we have the chain of equalities

$$\begin{aligned}
&\int_\Omega \left(\frac{\partial}{\partial t}\operatorname{grad}\delta\phi(\mathbf{r},\,t_1) + 4\pi\sigma\operatorname{grad}\delta\phi(\mathbf{r},\,t_1)\right)\operatorname{grad}\delta\phi(\mathbf{r},\,t_2)\,dV \\
&= \int_\Omega \operatorname{div}\left(\delta\phi(\mathbf{r},\,t_2)\left(\frac{\partial}{\partial t}\operatorname{grad}\delta\phi(\mathbf{r},\,t_1) + 4\pi\sigma\operatorname{grad}\delta\phi(\mathbf{r},\,t_1)\right)\right) dV \\
&\qquad\qquad - \int_\Omega \delta\phi(\mathbf{r},\,t_2)\operatorname{div}\left(\frac{\partial}{\partial t}\operatorname{grad}\delta\phi(\mathbf{r},\,t_1) + 4\pi\sigma\operatorname{grad}\delta\phi(\mathbf{r},\,t_1)\right) dV \\
&= \oint_{\Sigma_2} \delta\phi(\mathbf{r},\,t_2)\left(\frac{\partial}{\partial t}\operatorname{grad}\delta\phi(\mathbf{r},\,t_1) + 4\pi\sigma\operatorname{grad}\delta\phi(\mathbf{r},\,t_1)\right) d\mathbf{S} \\
&\qquad\qquad - \oint_{\Sigma_1} \delta\phi(\mathbf{r},\,t_2)\left(\frac{\partial}{\partial t}\operatorname{grad}\delta\phi(\mathbf{r},\,t_1) + 4\pi\sigma\operatorname{grad}\delta\phi(\mathbf{r},\,t_1)\right) d\mathbf{S} \\
&\qquad\qquad\qquad - \int_\Omega \delta\phi(\mathbf{r},\,t_2)\left(\frac{\partial}{\partial t}\Delta\delta\phi(\mathbf{r},\,t_1) + 4\pi\operatorname{div}\left(\sigma\operatorname{grad}\delta\phi(\mathbf{r},\,t_1)\right)\right) dV \\
&= \delta V_{\mathrm{i}}(t_2)\oint_{\Sigma_2}\left(\frac{\partial}{\partial t}\operatorname{grad}\delta\phi(\mathbf{r},\,t_1) + 4\pi\sigma\operatorname{grad}\delta\phi(\mathbf{r},\,t_1)\right) d\mathbf{S} \\
&= \delta V_{\mathrm{i}}(t_2)\oint_{\Sigma_1}\left(\frac{\partial}{\partial t}\operatorname{grad}\delta\phi(\mathbf{r},\,t_1) + 4\pi\sigma\operatorname{grad}\delta\phi(\mathbf{r},\,t_1)\right) d\mathbf{S} \\
&\qquad\qquad + \delta V_{\mathrm{i}}(t_2)\int_\Omega \operatorname{div}\left(\frac{\partial}{\partial t}\operatorname{grad}\delta\phi(\mathbf{r},\,t_1) + 4\pi\sigma\operatorname{grad}\delta\phi(\mathbf{r},\,t_1)\right) dV \\
&= \delta V_{\mathrm{i}}(t_2)\int_\Omega\left(\frac{\partial}{\partial t}\Delta\delta\phi(\mathbf{r},\,t_1) + 4\pi\operatorname{div}\left(\sigma\operatorname{grad}\delta\phi(\mathbf{r},\,t_1)\right)\right) dV = 0.
\end{aligned}$$

Setting $t_1 = t_2$, we get the equation

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left|\operatorname{grad}\delta\phi\right|^2\,dV + 4\pi\int_\Omega \sigma\left|\operatorname{grad}\delta\phi\right|^2\,dV = 0,$$

whence we obtain

$$\frac{d}{dt}\int_\Omega \left|\operatorname{grad}\delta\phi\right|^2\,dV \le 0.$$

Since $\delta\phi|_{t=0} = 0$, from this it follows that at any moment of time

$$\int_\Omega \left|\operatorname{grad}\delta\phi\right|^2\,dV = 0,$$

whereupon we find that $\operatorname{grad}\delta\phi = 0$. Since $\delta\phi|_{\Sigma_1} = 0$, this yields $\delta\phi = 0$, i.e. $\phi^{(1)} = \phi^{(2)}$ and $V_{\mathrm{i}}^{(1)} = V_{\mathrm{i}}^{(2)}$. Similar result for the stationary equations (16)–(18) can be obtained in the same fashion.

Thus one of the crucial aspects of spherical models of the GEC is that the value of the ionospheric potential can always be inferred from the solution. Another important idea is that the integral conditions (13) and (17) cannot be omitted, for they are direct consequences of Maxwell's equations and do not follow from the other equations for ϕ. If we set the ionospheric potential equal to a given value or function and omitted the integral conditions, the solutions to such a system of equations would not correspond to Maxwell's equations. Note that there can exist only one solution to (12)–(15) or (16)–(18), hence if the original Maxwell's equations have a solution (which we suppose to be the case), this solution is also the only solution to (12)–(15) or (16)–(18).

A more rigorous analysis employing Sobolev spaces and generalised functions makes it possible to prove that both problems (12)–(15) and (16)–(18) are well-posed—that is to say, if some restrictions are imposed on σ, $\mathbf{J}^{\mathrm{ext}}$, ϕ^0, Σ_1 and Σ_2, there exists a solution ϕ which is unique in a certain class of functions. Moreover, it can be shown that the solution of the non-stationary problem approaches the solution of the stationary problem as $t \to \infty$. Further references can be found in *Kalinin et al.* [2014].

Since the equations (12)–(15) and (16)–(18) are linear with respect to ϕ, the electric potential distributions satisfy the superposition principle. To be specific, let $\mathbf{J}^{\mathrm{ext}}_{(1)}(\mathbf{r})$ and $\mathbf{J}^{\mathrm{ext}}_{(2)}(\mathbf{r})$ be two external current density distributions and let $\phi^{(1)}(\mathbf{r})$ and $\phi^{(2)}(\mathbf{r})$ be the solutions to (16)–(18) with $\mathbf{J}^{\mathrm{ext}}_{(1)}$ and $\mathbf{J}^{\mathrm{ext}}_{(2)}$ respectively ($\sigma(\mathbf{r})$ is assumed to be the same in both cases). Then the electric potential distribution corresponding to $\mathbf{J}^{\mathrm{ext}}_{(1)} + \mathbf{J}^{\mathrm{ext}}_{(2)}$ is exactly $\phi^{(1)} + \phi^{(2)}$, and the ionospheric potential produced by $\mathbf{J}^{\mathrm{ext}}_{(1)} + \mathbf{J}^{\mathrm{ext}}_{(2)}$ is equal to the sum of the contributions from $\mathbf{J}^{\mathrm{ext}}_{(1)}$ and $\mathbf{J}^{\mathrm{ext}}_{(2)}$ calculated separately. This immediately follows from the linearity of the equations and the uniqueness of the solution: $\phi^{(1)} + \phi^{(2)}$, owing to the linearity of the equations, satisfies (16)–(18) with $\mathbf{J}^{\mathrm{ext}}_{(1)} + \mathbf{J}^{\mathrm{ext}}_{(2)}$ and therefore is the only solution. Similarly, let $\mathbf{J}^{\mathrm{ext}}_{(1)}(\mathbf{r},\, t)$ and $\mathbf{J}^{\mathrm{ext}}_{(2)}(\mathbf{r},\, t)$ be two external current density distributions, let $\phi^0_{(1)}(\mathbf{r})$ and $\phi^0_{(2)}(\mathbf{r})$ be two steady-state potential distributions and let $\phi^{(1)}(\mathbf{r},\, t)$ and $\phi^{(2)}(\mathbf{r},\, t)$ be the solutions to (12)–(15) with $\mathbf{J}^{\mathrm{ext}}_{(1)}$, $\phi^0_{(1)}$ and $\mathbf{J}^{\mathrm{ext}}_{(2)}$, $\phi^0_{(2)}$ respectively (with the same $\sigma(\mathbf{r},\, t)$). Then by a similar argument we obtain that the electric potential distribution corresponding to $\mathbf{J}^{\mathrm{ext}}_{(1)} + \mathbf{J}^{\mathrm{ext}}_{(2)}$ and $\phi^0_{(1)} + \phi^0_{(2)}$ is equal to $\phi^{(1)} + \phi^{(2)}$, and the ionospheric potential produced by $\mathbf{J}^{\mathrm{ext}}_{(1)} + \mathbf{J}^{\mathrm{ext}}_{(2)}$ is equal to the sum of the contributions from $\mathbf{J}^{\mathrm{ext}}_{(1)}$ and $\mathbf{J}^{\mathrm{ext}}_{(2)}$. These simple observations enable us to regard the net potential distribution and the total ionospheric potential as the sums of contributions from different thunderstorms.

A non-steady-state model

Henceforth we suppose that Σ_1 and Σ_2 are concentric spheres, their radii being equal to $r_{\min}$ and $r_{\max}$ respectively. Let $(r,\, \theta,\, \psi)$ be spherical coordinates whose origin coincides with the common centre of these spheres.

It can be shown that given the conductivity $\sigma(\mathbf{r},\, t)$, the external current density $\mathbf{J}^{\mathrm{ext}}(\mathbf{r},\, t)$ and the initial distribution of the potential $\phi^0(\mathbf{r})$, the potential distribution $\phi(\mathbf{r},\, t)$ corresponding to (12)–(15), and in particular the ionospheric potential, can always be calculated numerically by using the Galerkin method. Furthermore, it turns out that if the conductivity does not depend on θ and ψ and is of the form $\sigma(\mathbf{r},\, t) = \sigma(r,\, t)$, then it is possible to derive an explicit formula for the ionospheric potential $V_{\mathrm{i}}(t)$. Let us demonstrate this.

First of all, it easily follows from (12) and (13) that for all $h \in [r_{\min},\, r_{\max}]$

$$\oint_{\Sigma_h} \left(\frac{\partial}{\partial t} \operatorname{grad} \phi + 4\pi\sigma \operatorname{grad} \phi \right) d\mathbf{S} = 4\pi \oint_{\Sigma_h} \mathbf{J}^{\mathrm{ext}}\, d\mathbf{S},$$

Σ_h being the surface $\{\mathbf{r}\colon r = h\}$. In terms of $\mathbf{E} = -\operatorname{grad}\phi$ this equation reads as follows:

$$\int_0^{2\pi}\!\!\int_0^{\pi} \left(\frac{\partial E_r(r,\, \theta,\, \psi,\, t)}{\partial t} + 4\pi\sigma(r,\, t)\, E_r(r,\, \theta,\, \psi,\, t) \right) \sin\theta\, d\theta\, d\psi$$
$$= -4\pi \int_0^{2\pi}\!\!\int_0^{\pi} J^{\mathrm{ext}}_r(r,\, \theta,\, \psi,\, t) \sin\theta\, d\theta\, d\psi.$$

Introducing the notation

$$\bar{E}(r,\,t) = \frac{1}{4\pi}\int_0^{2\pi}\!\!\int_0^{\pi} E_r(r,\,\theta,\,\psi,\,t)\sin\theta\,d\theta\,d\psi, \qquad \bar{J}(r,\,t) = \frac{1}{4\pi}\int_0^{2\pi}\!\!\int_0^{\pi} J_r^{\mathrm{ext}}(r,\,\theta,\,\psi,\,t)\sin\theta\,d\theta\,d\psi,$$

we arrive at the equation

$$\frac{\partial \bar{E}(r,\,t)}{\partial t} + 4\pi\sigma(r,\,t)\,\bar{E}(r,\,t) = -4\pi\bar{J}(r,\,t)$$

with the initial condition

$$\bar{E}(r,\,0) = \frac{1}{4\pi}\int_0^{2\pi}\!\!\int_0^{\pi} E_r^0(r,\,\theta,\,\psi)\sin\theta\,d\theta\,d\psi,$$

where $\mathbf{E}^0 = -\operatorname{grad}\phi^0$. The general solution to this linear ordinary differential equation is given by the formula

$$\bar{E}(r,\,t) = \bar{E}(r,\,0)\exp\left(-4\pi\int_0^t \sigma(r,\,\tau)\,d\tau\right) - 4\pi\int_0^t \bar{J}(r,\,\tau)\exp\left(-4\pi\int_\tau^t \sigma(r,\,u)\,du\right)d\tau.$$

It is easy to ascertain that

$$V_{\mathrm{i}}(t) = -\int_{r_{\min}}^{r_{\max}} \bar{E}(r,\,t)\,dr,$$

from which we obtain the formula

$$\begin{aligned} V_{\mathrm{i}}(t) = &-\frac{1}{4\pi}\int_{r_{\min}}^{r_{\max}}\!\int_0^{2\pi}\!\!\int_0^{\pi} E_r^0(\mathbf{r})\exp\left(-4\pi\int_0^t \sigma(r,\,\tau)\,d\tau\right)\sin\theta\,d\theta\,d\psi\,dr \\ &+\int_0^t\!\int_{r_{\min}}^{r_{\max}}\!\int_0^{2\pi}\!\!\int_0^{\pi} J_r^{\mathrm{ext}}(\mathbf{r},\,\tau)\exp\left(-4\pi\int_\tau^t \sigma(r,\,u)\,du\right)\sin\theta\,d\theta\,d\psi\,dr\,d\tau. \end{aligned} \tag{19}$$

This formula expresses the ionospheric potential in terms of the external current density $\mathbf{J}^{\mathrm{ext}}$ and the initial electric field $\mathbf{E}^0$. A similar formula was obtained by *Morozov* [2005] for the case where the conductivity increases exponentially with altitude and the external current density distribution is restricted to a finite number of point current sources.

In case neither the conductivity, nor the external current density depend on t, the formula (19) can be written in a simpler form:

$$\begin{aligned} V_{\mathrm{i}}(t) = &-\frac{1}{4\pi}\int_{r_{\min}}^{r_{\max}}\!\int_0^{2\pi}\!\!\int_0^{\pi} E_r^0(\mathbf{r})\,e^{-4\pi\sigma(r)t}\sin\theta\,d\theta\,d\psi\,dr \\ &+\int_{r_{\min}}^{r_{\max}} \frac{1}{4\pi\sigma(r)}\int_0^{2\pi}\!\!\int_0^{\pi} J_r^{\mathrm{ext}}(\mathbf{r})\left(1-e^{-4\pi\sigma(r)t}\right)\sin\theta\,d\theta\,d\psi\,dr. \end{aligned} \tag{20}$$

A steady-state model

Let us now consider the stationary problem (16)–(18). As in the non-stationry case, given the conductivity $\sigma(\mathbf{r})$ and the external current density $\mathbf{J}^{\mathrm{ext}}(\mathbf{r})$, the potential distribution $\phi(\mathbf{r})$ corresponding to this problem, and in particular the ionospheric potential, can be calculated numerically by means of the Galerkin method, and again, if certain restrictions are imposed on the conductivity, we can express the ionospheric potential in terms of σ and $\mathbf{J}^{\mathrm{ext}}$. More precisely, the ionospheric potential can be described by an explicit formula, supposing that the conductivity is of the form $\sigma(\mathbf{r}) = a(r)\cdot b(\theta,\,\psi)$. Let us derive this formula.

As in the non-stationary case, we deduce from (16) and (17) that for all $h \in [r_{\min},\, r_{\max}]$

$$\oint_{\Sigma_h} \sigma \operatorname{grad} \phi \, d\mathbf{S} = \oint_{\Sigma_h} \mathbf{J}^{\mathrm{ext}} \, d\mathbf{S}$$

and reformulate this equation in terms of $\mathbf{E} = -\operatorname{grad} \phi$:

$$\int_0^{2\pi} \int_0^{\pi} a(r)\, b(\theta,\, \psi)\, E_r(r,\, \theta,\, \psi) \sin\theta \, d\theta \, d\psi = -\int_0^{2\pi} \int_0^{\pi} J_r^{\mathrm{ext}}(r,\, \theta,\, \psi) \sin\theta \, d\theta \, d\psi.$$

Dividing both parts of this equation by $a(r)$, integrating over r from $r_{\min}$ to $r_{\max}$ and using the obvious relation

$$V_{\mathrm{i}} = -\int_{r_{\min}}^{r_{\max}} E_r(r,\, \theta,\, \psi)\, dr,$$

we obtain the formula

$$V_{\mathrm{i}} = \frac{\displaystyle\int_{r_{\min}}^{r_{\max}} \frac{1}{a(r)} \int_0^{2\pi} \int_0^{\pi} J_r^{\mathrm{ext}}(\mathbf{r}) \sin\theta \, d\theta \, d\psi \, dr}{\displaystyle\int_0^{2\pi} \int_0^{\pi} b(\theta,\, \psi) \sin\theta \, d\theta \, d\psi}. \tag{21}$$

If the conductivity is a function of r alone, $\sigma(\mathbf{r}) = \sigma(r)$, the formula (21) can be simplified to

$$V_{\mathrm{i}} = \int_{r_{\min}}^{r_{\max}} \frac{1}{4\pi\sigma(r)} \int_0^{2\pi} \int_0^{\pi} J_r^{\mathrm{ext}}(\mathbf{r}) \sin\theta \, d\theta \, d\psi \, dr. \tag{22}$$

Note that (20) tends to (22) as $t \to \infty$.

PLANE-PARALLEL MODELS OF THE GEC

Early models of atmospheric electricity were developed within the framework of the ‘flat-Earth’ geometry, where the atmosphere is represented by a horizontally infinite plane-parallel slab. As the height of the atmosphere is much less than the Earth’s radius, such an approximation seems reasonable for calculating contributions from separate thunderstorms to the ionospheric potential. However, since spherical models of the GEC are more natural and have a number of advantages over ‘flat-Earth’ models (see below), here we confine ourselves only to the study of the simplest steady-state plane-parallel model.

Let $(x,\, y,\, z)$ be Cartesian coordinates. We suppose that the Earth’s surface Σ_1 is the plane $z = 0$ and the upper boundary of the atmosphere Σ_2 is the plane $z = L$. We assume that the conductivity $\sigma(\mathbf{r})$ is described by the function

$$\sigma(\mathbf{r}) = \sigma(z) = \sigma_0 \exp(z/H),$$

where σ_0 and H are certain constants, and we suppose that the external current density $\mathbf{J}^{\mathrm{ext}}(\mathbf{r})$ is non-zero only in a certain bounded region in the atmosphere. In accordance with the general theory, the steady-state distribution of the potential ϕ is described by the equations (16) and (18).

Let us first suppose that $\operatorname{div} \mathbf{J}^{\mathrm{ext}}(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}')$. Finding the solution to (16) in the infinite space and using the method of image sources modified for the exponential conductivity profile [*Kasemir*, 1959; *Willett*, 1979], one can show that one of the solutions to (16) in the region $\Omega = \{\mathbf{r}\colon 0 \le z \le L\}$ with the boundary conditions

$$\phi|_{\Sigma_1} = 0, \qquad \phi|_{\Sigma_2} = 0$$

is given by the function

$$G(\mathbf{r},\,\mathbf{r}') = \frac{\exp\left(-\dfrac{z+z'}{2H}\right)}{4\pi\sigma_0} \times$$
$$\sum\nolimits_{k=-\infty}^{+\infty} \left(\frac{\exp\left(-\dfrac{\sqrt{(z+z'-2kL)^2+(\boldsymbol{\rho}-\boldsymbol{\rho}')^2}}{2H}\right)}{\sqrt{(z+z'-2kL)^2+(\boldsymbol{\rho}-\boldsymbol{\rho}')^2}} - \frac{\exp\left(-\dfrac{\sqrt{(z-z'-2kL)^2+(\boldsymbol{\rho}-\boldsymbol{\rho}')^2}}{2H}\right)}{\sqrt{(z-z'-2kL)^2+(\boldsymbol{\rho}-\boldsymbol{\rho}')^2}} \right),$$

where $\mathbf{r} = (x,\,y,\,z)$, $\mathbf{r}' = (x',\,y',\,z')$, $\boldsymbol{\rho} = (x,\,y)$ and $\boldsymbol{\rho}' = (x',\,y')$.

In the general case of arbitrary $\operatorname{div}\mathbf{J}^{\mathrm{ext}}$, using the obtained $G(\mathbf{r},\,\mathbf{r}')$ as the Green's function of the equation (16), we obtain the solution

$$\Phi(\mathbf{r}) = \int_\Omega G(\mathbf{r},\,\mathbf{r}')\operatorname{div}\mathbf{J}^{\mathrm{ext}}(\mathbf{r}')\,dV',$$

which is equal to zero at both boundary surfaces. Making the substitution $\phi(\mathbf{r}) = \Phi(\mathbf{r}) + \psi(\mathbf{r})$, we obtain for ψ the equation

$$\operatorname{div}(\sigma\operatorname{grad}\psi) = 0 \tag{23}$$

with boundary conditions

$$\psi|_{\Sigma_1} = 0, \qquad \psi|_{\Sigma_2} = V_{\mathrm{i}}. \tag{24}$$

It is easy to verify that the problem (23), (24) has infinitely many solutions, unless we explicitly specify the value of the ionospheric potential. Indeed, for any constant C

$$\Psi(\mathbf{r},\,C) = C\,(1-\exp(-z/H))$$

is a solution to it, and the corresponding ionospheric potential is described by the formula

$$V_{\mathrm{i}} = C\,(1-\exp(-L/H)).$$

Here lies one of the principal disadvantages of plane-parallel models, the impossibility to uniquely determine the ionospheric potential from the equations without additional assumptions (see below), and from this it follows that the corresponding problem (16), (18) is ill-posed.

If we explicitly specify the value V_{i} of the potential at Σ_2, we can always find C_0 such that $\Psi(\mathbf{r},\,C_0)|_{\Sigma_2} = V_{\mathbf{i}}$, namely

$$C_0 = \frac{V_{\mathrm{i}}}{1-\exp(-L/H)}.$$

As before, making the substitution $\psi(\mathbf{r}) = \Psi(\mathbf{r},\,C_0) + \chi(\mathbf{r})$, we obtain for χ the equation

$$\operatorname{div}(\sigma\operatorname{grad}\chi) = 0 \tag{25}$$

with boundary conditions

$$\chi|_{\Sigma_1} = 0, \qquad \chi|_{\Sigma_2} = 0. \tag{26}$$

One of the solutions of the problem (25), (26) is $\chi = 0$, hence $\Phi(\mathbf{r}) + \Psi(\mathbf{r},\, C_0)$ is one of the solutions to the original equations (16) and (18) (see also *Ogawa* [1985]). Let us study this solution more closely. Since we have supposed that $\mathbf{J}^{\mathrm{ext}}(\mathbf{r}) \neq 0$ only in a certain bounded region in the atmosphere, we expect the net electric current I_Φ through Σ_2 corresponding to $\Phi(\mathbf{r})$ to be finite. In order to calculate this current, let us note that[‡]

$$\begin{aligned} I_\Phi = -\int_{\Sigma_2} \sigma(\mathbf{r})\,\mathrm{grad}\,\Phi(\mathbf{r})\,d\mathbf{S} &= -\int_{\Sigma_2} \sigma(\mathbf{r})\,\mathrm{grad}\left\{\int_\Omega G(\mathbf{r},\,\mathbf{r}')\,\mathrm{div}\,\mathbf{J}^{\mathrm{ext}}(\mathbf{r}')\,dV'\right\} d\mathbf{S} \\ &= -\int_\Omega \left\{\int_{\Sigma_2} \sigma(\mathbf{r})\,\mathrm{grad}\,G(\mathbf{r},\,\mathbf{r}')\,d\mathbf{S}\right\} \mathrm{div}\,\mathbf{J}^{\mathrm{ext}}(\mathbf{r}')\,dV'. \end{aligned}$$

A direct calculation shows that (see also *Willett* [1979]) for all $z' < L$

$$\int_{\Sigma_2} \sigma(\mathbf{r})\,\mathrm{grad}\,G(\mathbf{r},\,\mathbf{r}')\,d\mathbf{S} = \frac{1 - \exp\left(-z'/H\right)}{1 - \exp\left(-L/H\right)},$$

whence we obtain

$$\begin{aligned} I_\Phi &= -\int_\Omega \frac{1 - \exp\left(-z'/H\right)}{1 - \exp\left(-L/H\right)}\,\mathrm{div}\,\mathbf{J}^{\mathrm{ext}}(\mathbf{r}')\,dV' \\ &= -\int_\Omega \mathrm{div}\left(\frac{1 - \exp\left(-z'/H\right)}{1 - \exp\left(-L/H\right)}\,\mathbf{J}^{\mathrm{ext}}(\mathbf{r}')\right) dV' + \int_\Omega \mathbf{J}^{\mathrm{ext}}(\mathbf{r}')\,\mathrm{grad}\,\frac{1 - \exp\left(-z'/H\right)}{1 - \exp\left(-L/H\right)}\,dV'. \end{aligned}$$

Let us suppose that $\mathbf{J}^{\mathrm{ext}}\big|_{\Sigma_2} = 0$; then the first summand is equal to zero, and therefore

$$I_\Phi = \int_\Omega J_z^{\mathrm{ext}}(\mathbf{r}')\,\frac{1}{H}\,\frac{\exp\left(-z'/H\right)}{1 - \exp\left(-L/H\right)}\,dV' = \frac{\sigma_0}{H\left(1 - \exp\left(-L/H\right)\right)} \int_\Omega \frac{J_z^{\mathrm{ext}}(\mathbf{r}')}{\sigma(z')}\,dV'.$$

Thus the net electric current through the surface Σ_2 corresponding to the potential distribution $\Phi(\mathbf{r})$ is finite, as we have expected. Obviously, a similar integral describing its counterpart corresponding to $\Psi(\mathbf{r},\, C_0)$ does not converge, but we can employ the spherical geometry of the real atmosphere to surmount this difficulty. If we restrict the region of integration such that its area will be equal to the area of the (spherical) Earth's surface, the net electric current I_Ψ through Σ_2 corresponding to $\Psi(\mathbf{r},\, C_0)$ will be equal to

$$\begin{aligned} I_\Psi &= -4\pi r_0^2\,\left(\sigma(\mathbf{r})\,\mathrm{grad}\,\Psi(\mathbf{r},\, C_0)\right)\big|_{\Sigma_2} \\ &= -4\pi r_0^2\,\sigma_0 \exp\left(L/H\right)\mathrm{grad}\,\left.\frac{V_{\mathrm{i}}\left(1 - \exp\left(-z/H\right)\right)}{1 - \exp\left(-L/H\right)}\right|_{z=L} = -\frac{4\pi r_0^2\,V_{\mathrm{i}}\,\sigma_0}{H\left(1 - \exp\left(-L/H\right)\right)}, \end{aligned}$$

r_0 standing for the Earth's radius. If we demand that the two currents I_Φ and I_Ψ compensate for one another, i.e. $I_\Phi + I_\Psi = 0$, then we immediately obtain the equation

$$V_{\mathrm{i}} = \frac{1}{4\pi r_0^2} \int_\Omega \frac{J_z^{\mathrm{ext}}(\mathbf{r}')}{\sigma(z')}\,dV'. \tag{27}$$

Obviously, the ionospheric potential given by the formula (27) is close to the exact value given by (22). Therefore it seems reasonable to expect that χ in $\phi(\mathbf{r}) = \Phi(\mathbf{r}) + \Psi(\mathbf{r},\, C_0) + \chi(\mathbf{r})$ is equal to zero. However, it is not difficult to show that unless we specify the boundary conditions at infinity (when

[‡]In this equation the gradient is taken with respect to $\mathbf{r}$ and the divergence is taken with respect to $\mathbf{r}'$.

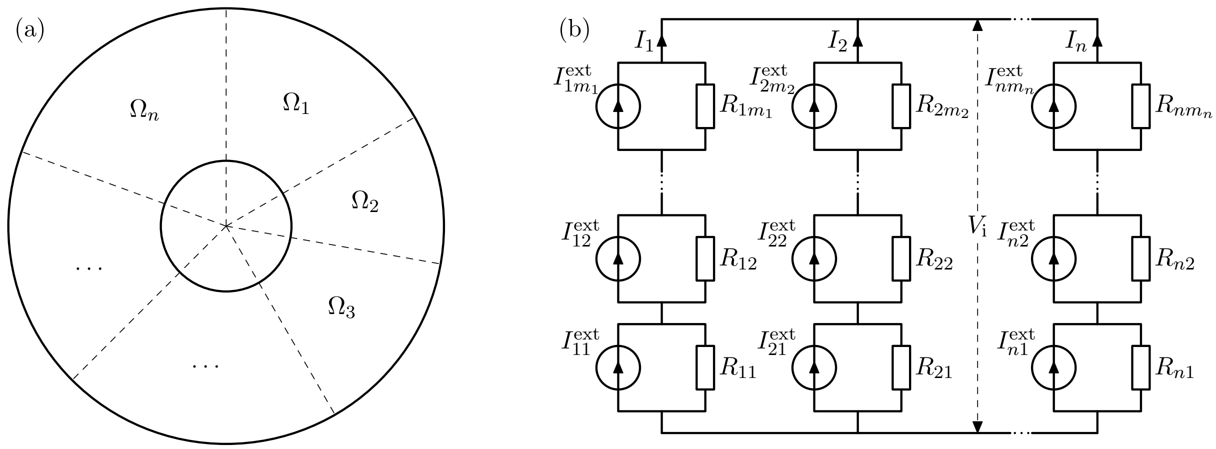

Figure 2: (a) The partition of the atmosphere. (b) The equivalent electric circuit.

$|\boldsymbol{\rho}| \to \infty$), the problem (25), (26) has non-trivial solutions. It is possible to impose such restrictions on the behaviour of χ at infinity that the solution $\chi = 0$ will be unique; in spite of that, here we will refrain from further development of this idea, for it is unnatural to establish the boundary conditions for the function $\chi(\mathbf{r}) = \phi(\mathbf{r}) - \Phi(\mathbf{r}) - \Psi(\mathbf{r},\, C_0)$ instead of the original potential $\phi(\mathbf{r})$. We can conclude that although the plane-parallel approximation seems reasonable for certain problems, it has a number of disadvantages, since there are considerable difficulties with establishing boundary conditions at infinity, the corresponding problem cannot easily be made well-posed and the ionospheric potential cannot be uniquely determined from the model equations without additional artificial assumptions. Therefore in most cases models employing spherical geometry seem more natural and convenient.

APPROXIMATE ANALYSIS AND ITS CONNECTION WITH CLASSICAL MULTI-COLUMN MODELS OF THE GEC

Let us now consider the steady-state problem (16)–(18) (in the spherical geometry). Although the ionospheric potential can always be calculated numerically, it is nevertheless convenient to be able to express it analytically by means of (21), even though it requires the conductivity to be of the form $\sigma(\mathbf{r}) = a(r) \cdot b(\theta,\, \psi)$. In this section we will show that it is possible to derive an approximate formula for the ionospheric potential in another special case, where weaker restrictions are imposed on the conductivity, and, what is more, the corresponding approximation turns out to be a generalisation of classical multi-column models of atmospheric electricity based on the concept of the equivalent electric circuit.

The region occupied by the atmosphere being denoted by Ω, we can write $\Omega = \Gamma \times [r_{\min},\, r_{\max}]$, where Γ is the unit sphere and '$\times$' indicates the Cartesian product. Suppose that

$$\Gamma = \bigcup\nolimits_{j=1}^{n} \Gamma_j$$

and $\Gamma_i \cap \Gamma_j = \varnothing$ for $i \neq j$. This partition of Γ induces a corresponding partition of Ω, namely

$$\Omega = \bigcup\nolimits_{j=1}^{n} \Omega_j,$$

where $\Omega_j = \Gamma_j \times [r_{\min},\, r_{\max}]$ (an example of such a partition is shown schematically in Fig. 2a). Using the spherical coordinates introduced before, let us suppose that $\sigma(\mathbf{r})$ and $J_r^{\mathrm{ext}}(\mathbf{r})$ are functions of r alone within

Ω_j, and $J^{\mathrm{ext}}_{\theta,\,\psi}(\mathbf{r}) = 0$ in Ω. More precisely, we suppose that

$$\sigma(r,\,\theta,\,\psi) = \begin{cases} \sigma^{(1)}(r), & (r,\,\theta,\,\psi) \in \Omega_1, \\ \sigma^{(2)}(r), & (r,\,\theta,\,\psi) \in \Omega_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots & \\ \sigma^{(n)}(r), & (r,\,\theta,\,\psi) \in \Omega_n, \end{cases}$$

$$J^{\mathrm{ext}}_r(r,\,\theta,\,\psi) = \begin{cases} J^{(1)}_r(r), & (r,\,\theta,\,\psi) \in \Omega_1, \\ J^{(2)}_r(r), & (r,\,\theta,\,\psi) \in \Omega_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots & \\ J^{(n)}_r(r), & (r,\,\theta,\,\psi) \in \Omega_n \end{cases}$$

and

$$J^{\mathrm{ext}}_{\theta,\,\psi}(r,\,\theta,\,\psi) = 0 \quad \text{for all } (r,\,\theta,\,\psi).$$

Let us suppose that these conditions are satisfied, and let us suppose that, for all j, the characteristic 'horizontal' scale L_j of Ω_j is much greater than the characteristic vertical scale $R = r_{\max} - r_{\min}$, in which case the derivatives with respect to θ and ψ in (16) can be neglected. Then within Ω_j we get

$$\frac{\partial}{\partial r}\left(r^2\sigma^{(j)}(r)\frac{\partial\phi(r,\,\theta,\,\psi)}{\partial r}\right) = \frac{d}{dr}\left(r^2 J^{(j)}_r(r)\right),$$

whence by integration over r we obtain

$$\frac{\partial\phi(r,\,\theta,\,\psi)}{\partial r} = \frac{J^{(j)}_r(r)}{\sigma^{(j)}(r)} + \frac{C_j(\theta,\,\psi)}{r^2\sigma^{(j)}(r)}, \tag{28}$$

$C_j(\theta,\,\psi)$ being a certain function. Therefore for all θ and ψ we have

$$V_{\mathrm{i}} = \int_{r_{\min}}^{r_{\max}} \frac{\partial\phi(r,\,\theta,\,\psi)}{\partial r}\,dr = \int_{r_{\min}}^{r_{\max}} \frac{J^{(j)}_r(r)\,dr}{\sigma^{(j)}(r)} + C_j(\theta,\,\psi)\int_{r_{\min}}^{r_{\max}} \frac{dr}{r^2\sigma^{(j)}(r)}, \tag{29}$$

from which it follows that C_j is actually a constant, independent of θ and ψ. With j in the range $1 \le j \le n$, it gives n equations in $n+1$ variables $C_1,\,C_2,\,\dots,\,C_n$ and V_{i}. The equation which closes the system is obtained by substitution of (28) into (17), which gives

$$\sum\nolimits_{j=1}^{n} \gamma_j C_j = 0, \tag{30}$$

γ_j being the solid angle subtended by Γ_j (and thus the sum of γ_j being equal to 4π). Eliminating all C_j from (29) and (30), we obtain the following formula for the ionospheric potential:

$$V_{\mathrm{i}} = \sum\nolimits_{j=1}^{n} \frac{\gamma_j \displaystyle\int_{r_{\min}}^{r_{\max}} \frac{J^{(j)}_r(r)\,dr}{\sigma^{(j)}(r)}}{\displaystyle\int_{r_{\min}}^{r_{\max}} \frac{dr}{r^2\sigma^{(j)}(r)}} \Bigg/ \sum\nolimits_{j=1}^{n} \frac{\gamma_j}{\displaystyle\int_{r_{\min}}^{r_{\max}} \frac{dr}{r^2\sigma^{(j)}(r)}}. \tag{31}$$

It is easy to see that for $n = 1$ the approximate formula (31) coincides with the exact expression (22). Futhermore, in case the conductivity is of the form $\sigma(\mathbf{r}) = a(r)\cdot b(\theta,\,\psi)$, this formula turns into (21), provided that the regions Ω_j are taken to be infinitesimally small. In order to show this, let us point out that if $b(\theta,\, \psi)$ is a piecewise constant function, being equal to b_j within Ω_j, then (31) can be simplified to

$$V_\mathrm{i} = \frac{\displaystyle\int_{r_\mathrm{min}}^{r_\mathrm{max}} \frac{1}{a(r)} \sum\nolimits_{j=1}^{n} \gamma_j J_r^{(j)}(r)\, dr}{\sum\nolimits_{j=1}^{n} \gamma_j b_j},$$

which turns into (21) as $\gamma_j \to 0$, for this allows us to replace the summation over j by integration of continuous functions over θ and ψ.

As it has been stated before, such an approximation turns out to be a generalisation of classical multi-column models of atmospheric electricity based on the idea of the equivalent circuit. In such models the entire atmosphere is divided into two or more columns, some corresponding to thunderstorm regions, where the current flows upwards, and others corresponding to fair weather regions, where the current flows downwards. Replacing different regions with equivalent resistors and current sources, it is possible to simulate the real atmospheric electric system by an equivalent circuit. Such a circuit generalising those used in models by *Willett* [1979], *Volland* [1984] and *Odzimek et al.* [2010] is shown in Fig. 2b. It consists of n parallel vertical current paths whose lower ends are joined together, as are their upper ends. The jth vertical current path is supposed to be a series of infinitesimal elements consisting of a current source of strength I_{jk}^ext and a resistor of resistance R_{jk} connected in parallel.

To establish the correspondence between this circuit and the spherical model studied above, we say that n vertical current paths correspond to the regions $\Omega_1,\, \Omega_2,\, \ldots,\, \Omega_n$, the bottom and top of the circuit representing the boundary surfaces Σ_1 and Σ_2 respectively. We also demand that the resistances R_{jk} and the external currents I_{jk}^ext correspond to the distributions $\sigma^{(j)}(r)$ and $J_r^{(j)}(r)$. More precisely, we suppose that the region $\Omega_j = \Gamma_j \times [r_\mathrm{min},\, r_\mathrm{max}]$ is partitioned into infinitesimally thin slabs $\Gamma_j \times [r_{jk},\, r_{j,\,k+1}]$ with $r_{j,\,k+1} - r_{jk} = dr_{jk}$, and the kth slab corresponds to the element with the resistance R_{jk} and the external current I_{jk}^ext, i.e.

$$R_{jk} = \frac{dr_{jk}}{\gamma_j r_{jk}^2 \sigma^{(j)}(r_{jk})}, \qquad I_{jk}^\mathrm{ext} = \gamma_j r_{jk}^2 J_r^{(j)}(r_{jk}). \tag{32}$$

It is easy to see that the two approaches are actually equivalent, inasmuch as in either case we divide the atmosphere into one-dimensional columns and neglect the current flowing through their side surfaces.

Let us calculate the ionospheric potential using the equivalent circuit representation. Denoting the current in the jth path as I_j (which is defined to be positive if the current flows upwards and negative otherwise), we get n equations of the form

$$V_\mathrm{i} = \sum\nolimits_k \big(I_{jk}^\mathrm{ext} - I_j\big) R_{jk} = \sum\nolimits_k I_{jk}^\mathrm{ext} R_{jk} - I_j \sum\nolimits_k R_{jk},$$

where V_i stands for the voltage between the top and bottom of the circuit, thus being an equivalent of the ionospheric potential. Then, since

$$\sum\nolimits_{j=1}^{n} I_j = 0,$$

we obtain the formula

$$V_\mathrm{i} = \sum\nolimits_{j=1}^{n} \frac{\sum\nolimits_k I_{jk}^\mathrm{ext} R_{jk}}{\sum\nolimits_k R_{jk}} \bigg/ \sum\nolimits_{j=1}^{n} \frac{1}{\sum\nolimits_k R_{jk}}, \tag{33}$$

which is similar to that obtained by *Odzimek et al.* [2010]. As one might have expected, substituting the relations (32) into (33) and replacing the sums over k with integrals over r, we obtain the formula (31) again.

CONCLUSIONS

In order to compare spherical and plane-parallel models of the GEC, we have analysed both approaches, starting from the general Maxwell's equations. Since 'flat-Earth' models of the GEC turn out to be ill-posed without additional artificial assumptions, we can conclude that the spherical geometry is more natural and convenient for modelling the GEC. Both stationary and quasi-stationary spherical models correspond to well-posed problems, in which the ionospheric potential can be uniquely determined from the solution. The possibility to find the ionospheric potential together with the importance of integral conditions (13) and (17) are the most crucial aspects of spherical models of atmospheric electricity.

In certain simple cases the ionospheric potential in spherical models can be expressed analytically in terms of the conductivity, the external current density and the initial distribution of the electric field (in the non-stationary case). Since plane-parallel and multi-column models of the GEC can be regarded as approximations of spherical models, they also enable us to derive several approximate formulae for the ionospheric potential. Analytical expressions for the ionospheric potential can be used, for example, for parameterisation of atmospheric electricity in high-resolution weather and climate models.

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