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Potential energy for slantwise parcel motion

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Summary: Two formulations for the potential energy for slantwise motion are compared: one which applies strictly only to two-dimensional flows (SCAPE) and a three-dimensional formulation based on a Bernoulli equation. The two formulations share an identical contribution from the vertically integrated buoyancy anomaly and a contribution from different Coriolis terms. The latter arise from the neglect of (different) components of the total change in kinetic energy along a trajectory in the two formulations. This neglect is necessary in order to quantify the potential energy available for slantwise motion relative to a defined steady environment.

Keywords: Potential energy; slantwise; SCAPE.

1. INTRODUCTION

Parcel theory, in which the atmosphere is considered as the sum of an environment and a perturbation to that environment, can be used to quantify the potential energy available for conversion into kinetic energy. For quasi-vertical parcel ascent, and motion rapid enough that the effect of the Coriolis force can be considered negligible, this leads to the familiar concept of Convective Available Potential Energy (CAPE) (Moncrieff and Miller, 1976).

There have been two 'extensions' of the CAPE concept to allow consideration of slantwise ascent which, by its nature, is sufficiently slow that the rotation of the Earth plays a significant role. The first, which predates the introduction of CAPE, is due to Green *et al.* (1966) who refer to it as an 'extended' parcel theory and the second is known as Slantwise Convective Available Potential Energy (SCAPE) (Emanuel, 1983).

Slantwise convection, which occurs if conditional symmetric instability (CSI) is released, has been proposed to contribute to the formation of frontal rainbands (Bennetts and Hoskins, 1979) and the cloud heads which are often associated with explosively deepening cyclones (Shutts, 1990). SCAPE is a measure of the degree to which the atmosphere is unstable to CSI and is defined as the change of

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kinetic energy due to the momentum components transverse to the geostrophic flow. This theory can strictly only be applied to two-dimensional, steady state flows and further assumptions, not described here, must be made to apply it to more realistic three-dimensional evolving flows (Gray and Thorpe, 2000).

Prior to the development of the theory for CSI, Green *et al.* (1966) proposed their extended parcel theory for slantwise ascent through cloud systems, such as a major cold front, which they assumed to be in a steady state. They considered pseudo-adiabatic flow along slantwise trajectories and used an atmospheric application of Bernoulli's equation to obtain an expression for the change in kinetic energy. They found good agreement between observed and calculated windspeeds for the majority of sounding stations along an inferred parcel trajectory. Betts and McIlveen (1969) later proposed a correction to this energy formula for a system moving steadily relative to the Earth's surface. This formula can be applied to three-dimensional flows without simplification and hence appears preferable to that of SCAPE. The aim of this letter is to clarify the relationship between these two theories by re-examining in detail the assumptions made.

2. DERIVATION OF SCAPE

Consider a two-dimensional baroclinic zone orientated parallel to the y-axis such that the components of the parcel velocity, $\mathbf{v}_p$, are $v_p = v_e + v'$, $u_p = u'$ and $w_p = w'$ where $'$ denotes an anomaly and the environment, denoted by the subscript e, is in geostrophic balance. The Boussinesq momentum equations can be written:

$$\frac{Du'}{Dt} - fv' = 0, \tag{1}$$

$$\frac{Dv_p}{Dt} + fu_p = 0, \tag{2}$$

$$\frac{Dw'}{Dt} = g\left(\frac{\theta_p - \theta_e}{\theta_e}\right), \tag{3}$$

where f is the Coriolis parameter (considered constant), g is the acceleration due to gravity, θ is the potential temperature, and it has been assumed that the environment is in hydrostatic balance, that the ideal gas law applies, and that the perturbation pressure forces acting on the moving parcel can be neglected. The equation for the v component can be rewritten as:

$$\frac{Dv_p}{Dt} + f\,\frac{Dx}{Dt} = \frac{DM_p}{Dt} = 0, \tag{4}$$

where the pseudo-angular momentum, $M_p = v_p + fx$, is conserved following the parcel. SCAPE is defined as the change in kinetic energy along a trajectory, normally between a parcel origin and neutral buoyancy position, arising from the momentum components transverse to the geostrophic flow yielding:

$$\begin{aligned}\text{SCAPE} &= \int \left[\frac{D}{Dt}\left(\frac{1}{2}w'^2\right) + \frac{D}{Dt}\left(\frac{1}{2}u'^2\right)\right] dt, \\ &= \int \left[w' g \left(\frac{\theta_p - \theta_e}{\theta_e}\right) + u' f v' \right] dt, \\ &= \int g\left(\frac{\theta_p - \theta_e}{\theta_e}\right) dz + \int f(M_p - M_e)\, dx. \end{aligned} \tag{5}$$

3. DERIVATION OF THE FORMULATION OF GREEN *ET AL.*

Green *et al.* derive an atmospheric form of Bernoulli's equation for adiabatic slantwise ascent using the three-dimensional form of the momentum equation (ignoring friction) and the first law of thermodynamics. The flow is decomposed into a steady system velocity, **c**, and a system relative velocity, $\mathbf{u}_r$. Note that the velocity relative to the system includes both the perturbation velocity and the spatially varying part of the environmental velocity defined in Section 2. The change in kinetic energy along the trajectory is obtained by subtracting a Bernoulli equation for vertical pseudo-adiabatic ascent in a horizontally uniform hydrostatic environment from the Bernoulli equation for slantwise ascent, where the same changes in states occur along both trajectories. Betts and McIlveen derive a 'correction' term to the energy expression, neglected by Green *et al.*, due to the moving system.

The process of subtracting a horizontally uniform hydrostatic environment from the momentum equations is the basis of the Boussinesq approximation in which the pressure and potential temperature are partitioned such that $p = \bar{p}(z) + \delta p(x, y, z, t)$ and $\theta = \bar{\theta}(z) + \delta\theta(x, y, z, t)$ where $\bar{p}$ and $\bar{\theta}$ define a reference state. The expression for the change in kinetic energy along a slantwise trajectory derived by Green *et al.* and Betts and McIlveen can be derived more simply by beginning from the Boussinesq form of the three-dimensional momentum equation:

$$\frac{D\mathbf{u}_p}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{u}_p = -\nabla\left(\frac{\delta p}{\bar{\rho}}\right) + g\,\frac{\delta\theta}{\bar{\theta}}\,\hat{\mathbf{k}}, \tag{6}$$

where $\boldsymbol{\Omega}$ is the angular velocity of the rotating Earth. The correction term derived by Betts and McIlveen arises through consideration of motion relative to a frame moving with the system velocity. Betts and McIlveen consider a frame rotating with constant angular velocity, $\boldsymbol{\Omega}_f$, relative to the rotating Earth. The horizontal acceleration relative to an inertial frame can be written as:

$$\frac{D\mathbf{u}_r}{Dt_r} + 2(\boldsymbol{\Omega} + \boldsymbol{\Omega}_f) \times \mathbf{u}_r + 2\boldsymbol{\Omega} \times (\boldsymbol{\Omega}_f \times \mathbf{R}) + \boldsymbol{\Omega}_f \times (\boldsymbol{\Omega}_f \times \mathbf{R}) + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{R}), \tag{7}$$

where D/Dt_r is the time derivative following the system relative motion, and the system velocity, $\mathbf{c}$, is given by $\mathbf{\Omega}_f \times \mathbf{R}$. The first four terms of (7) are equivalent to the left-hand-side of (6) and the last term is the familiar centripetal acceleration which is incorporated into the apparent gravitational acceleration. In the formulation of Green *et al.*, the kinetic energy is given by the time integral of $\mathbf{u}_r$·(6) using the acceleration term in (7). Hence, the second term in (7) disappears on formulating the energy expression. The fourth term is negligible in comparison to the third.

The change in kinetic energy along a trajectory is thus given by:

$$\int \frac{D}{Dt_r}\left(\frac{1}{2}\mathbf{u}_r^2\right)dt = \int\left(w_r g\frac{\delta\theta}{\bar{\theta}}\right)dt - \int \mathbf{u}_r \cdot (f\hat{\mathbf{k}} \times \mathbf{c})\,dt - \int\left[\mathbf{u}_r \cdot \nabla\left(\frac{\delta p}{\bar{\rho}}\right)\right]dt, \tag{8}$$

in which $2\mathbf{\Omega}$ has been simplified to $f\hat{\mathbf{k}}$. The first term on the right-hand-side of (8) is a vertically integrated buoyancy anomaly term, effectively identical to that in (5), although a higher level of approximation is invoked by the Boussinesq approach followed here than by the parcel theory used in the SCAPE formulation (note that $w_r = w'$ for a hydrostatically balanced environment). The buoyancy term is written as $g(\Delta Z' - \Delta Z)$ in Green *et al.* where ΔZ and $\Delta Z'$ are the thicknesses between two isobaric surfaces along the slantwise and pseudo-adiabatic vertical trajectories respectively. Using hydrostatic balance, this can be written in integral form as $\int R(T_p - T_e)d\ln p$ which is mathematically equivalent to the buoyancy term in (5) (see Emanuel (1994) for further elaboration). The second term on the right-hand-side of the (8) is the correction term derived by Betts and McIlveen. The third term does not appear explicitly in the Green *et al.* derivation and arises from subtraction of the Bernoulli equation for vertical pseudo-adiabatic ascent in a horizontally uniform hydrostatic environment from that for slantwise ascent. The dynamical significance of the second and third terms is more apparent if, as in the SCAPE formulation, the environment flow, $\mathbf{u}_e$, is assumed to be in geostrophic balance and the vertical gradient of the perturbation pressure is neglected (a common assumption of parcel theory). If made here, this simplification yields,

$$\int \frac{D}{Dt_r}\left(\frac{1}{2}\mathbf{u}_r^2\right)dt = \int\left(w_r g\frac{\delta\theta}{\bar{\theta}}\right)dt - \int \mathbf{u}_r \cdot (f\hat{\mathbf{k}} \times \mathbf{u}')\,dt, \tag{9}$$

after some algebra. This can be written in the form

$$\int \frac{D}{Dt_r}\left(\frac{1}{2}\mathbf{u}_r^2\right)dt = \int\left(g\frac{\delta\theta}{\bar{\theta}}\right)dz - \int (f\mathbf{k}' \times \mathbf{u}')\cdot d\mathbf{x}, \tag{10}$$

where $\mathbf{x}$ is the distance vector in the reference frame moving with the system velocity. If the system velocity is in the direction of the geostrophic flow, then this reduces to the expression for SCAPE if only the energy arising from the momentum components transverse to the geostrophic flow are considered. Thus the two formulations are then consistent for two-dimensional geostrophic flow.

4. COMPARATIVE ANALYSIS

Definition of potential energy

The potential energy available to a parcel travelling along a slantwise trajectory is defined differently in the SCAPE and Green *et al.* formulations. The total kinetic energy per unit mass of a flow relative to the rotating Earth is given by $\frac{1}{2}\mathbf{u}_p^2$. SCAPE is defined as the change in kinetic energy along a trajectory arising from the transverse motion *only*, i.e. $\frac{1}{2}(u'^2 + w'^2)$.

In the formulation of Green *et al.*, the potential energy available to a parcel is defined as $\int D/Dt_r\left(\frac{1}{2}\mathbf{u}_r^2\right)dt$, which is the change in kinetic energy along a trajectory relative to a frame moving with the system velocity, $\mathbf{c}$.

Comparison of terms

The SCAPE and Green *et al.* formulations for potential energy share an identical contribution from a vertically integrated buoyancy anomaly, usually termed CAPE, and a contribution from different Coriolis terms. The Coriolis force does no work. In the SCAPE derivation the Coriolis term only appears due to the neglect of the contribution to the change in kinetic energy along a trajectory from the momentum component in the direction of the geostrophic flow. If SCAPE were instead defined in terms of a three-dimensional kinetic energy, e.g. as $\int[D(\frac{1}{2}\mathbf{u}'^2)/Dt]dt$, then the additional $\int(v'Dv'/Dt)dt$ term would include a term $\int -fu'v'dt$, which cancels the energy obtained from the Coriolis term in the equation for SCAPE. The geostrophic flow is effectively treated as an 'infinite reservoir' of kinetic energy from which energy can be transferred to the transverse circulations without cost.

In the formulation of Green *et al.* the Coriolis term appears due to the neglect of a different contribution to the change in kinetic energy along a trajectory. Expanding the total change in kinetic energy along a trajectory considered in a frame moving with the rotating Earth, $\int D/Dt\left(\frac{1}{2}\mathbf{u}_p^2\right)dt$, using $\mathbf{u}_p = \mathbf{c} + \mathbf{u}_r$ where $\mathbf{c}$ is constant,[1] yields

$$\int \frac{D}{Dt}\left(\frac{1}{2}\mathbf{u}_p^2\right)dt = \int\left(\frac{D}{Dt_r}\frac{1}{2}\mathbf{u}_r^2\right)dt + \int\left(\mathbf{c}\cdot\frac{D}{Dt_r}\mathbf{u}_r\right)dt. \tag{11}$$

Using (6) and (7), it can be shown that

$$\mathbf{c}\cdot\frac{D}{Dt_r}\mathbf{u}_r = -\mathbf{c}\cdot\nabla\left(\frac{\delta p}{\bar{\rho}}\right) - \mathbf{c}\cdot(f\hat{\mathbf{k}}\times\mathbf{u}_r), \tag{12}$$

and substituting (12) and (8) into (11) yields

$$\int \frac{D}{Dt}\left(\frac{1}{2}\mathbf{u}_p^2\right)dt = \mathrm{CAPE} - \mathbf{u}_p\cdot\nabla\left(\frac{\delta p}{\bar{\rho}}\right), \tag{13}$$

[1] Note that the derived energy expression is independent of whether the system velocity or *angular* velocity is assumed constant since the only effect on (7) is in the second term which does not contribute to the energy expression.

in which the pressure gradient term vanishes for geostrophic flow. This also follows directly from (6) and shows how the Coriolis term disappears when considering the total change in kinetic energy along a trajectory.

5. CONCLUSIONS

Two formulations for the potential energy for slantwise motion have been described and contrasted in this letter. Both formulations have two contributions to the energy. One is a contribution from a vertically intergrated buoyancy anomaly, usually termed CAPE, and is identical in the two formulations. The other is a contribution arising from a Coriolis term and is different in the two formulations. These latter terms only appear due to the neglect of (different) components of the total change in kinetic energy along a trajectory. The neglect of these components of the total kinetic energy is necessary to quantify the energy available for slantwise motion relative to a defined steady environment; motion transverse to the geostrophic flow in the SCAPE formulation and motion relative to the system motion in the Green *et al.* formulation. Such quantification is useful to assess the ability of the flow to extract energy from this steady environment in which it is embedded. Thus, SCAPE, for example, is not a unique definition of the energy available for slantwise motions and in situations where a uniform system speed can be defined the version of the Green *et al.* formulation presented here may be more appropriate and useful.

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REFERENCES

Bennetts, D. A. and Hoskins, B. J., 1979. Conditional symmetric instability—a possible explanation for frontal rainbands. *Q. J. R. Meteorol. Soc.*, **105**, 945–962.

Betts, A. K. and McIlveen, J. F. R., 1969. The energy formula in a moving reference frame. *Q. J. R. Meteorol. Soc.*, **95**, 639–642.

Emanuel, K. A., 1983. The Lagrangian parcel dynamics of moist symmetric instability. *J. Atmos. Sci.*, **40**, 2368–2376.

Emanuel, K. A., 1994. *Atmospheric Convection*. Oxford: Oxford University Press.

Gray, S. L. and Thorpe, A. J., 2000. Parcel theory in three-dimensions and the calculation of SCAPE. *Mon. Wea. Rev*, in press.

Green, J. S. A., Ludlam, F. H. and McIlveen, J. F. R., 1966. Isentropic flow analysis and the parcel theory. *Q. J. R. Meteorol. Soc.*, **92**, 210–219.

Moncrieff, M. W. and Miller, M. J., 1976. The dynamics and simulation of tropical cumulonimbus and squall lines. *Q. J. R. Meteorol. Soc.*, **120**, 373–394.

Shutts, G. J., 1990. SCAPE charts from numerical weather prediction model fields. *Mon. Wea. Rev*, **118**, 2745–2751.