Numerical Weather Prediction

In about a week
(3 lectures)

A bit of history....

- NWP was born at the Institute for Advanced Study in Princeton in 1940's - first electronic computer
- Since then, NWP has been one of the heaviest users of supercomputers.

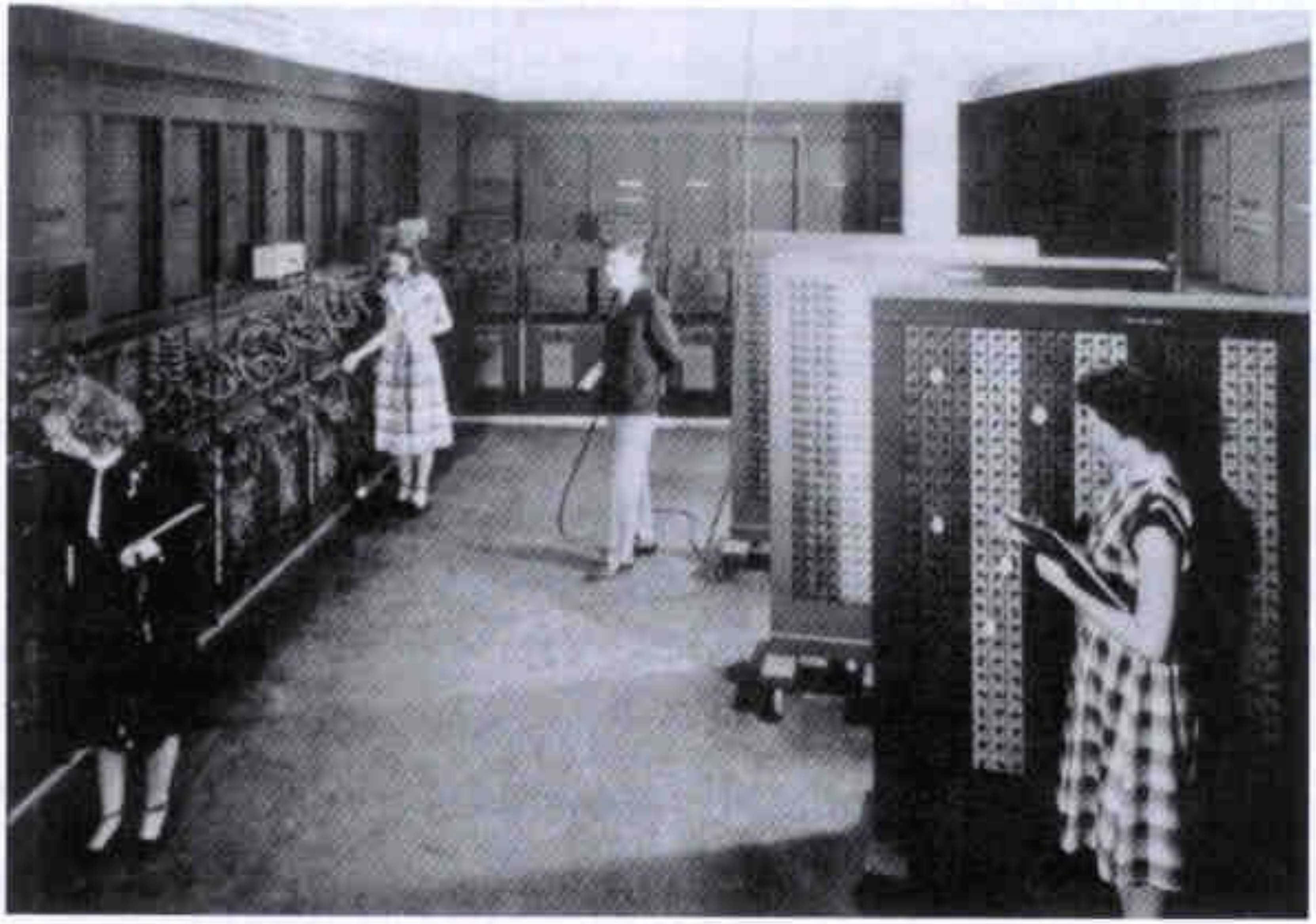
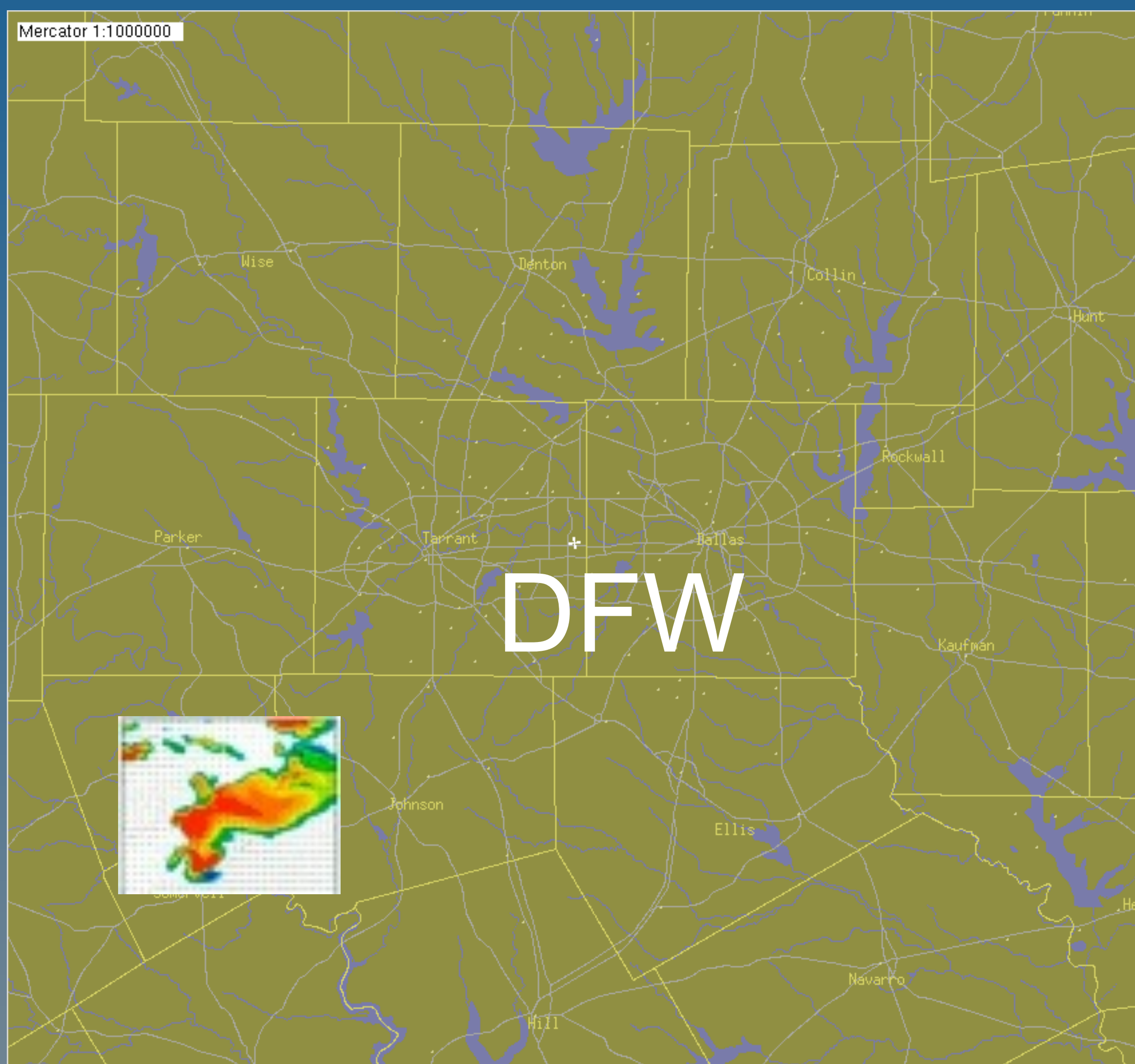


Figure 3: The ENIAC computer in 1948. The operators are changing the plug-in wiring. (PLATZMAN, 1979).

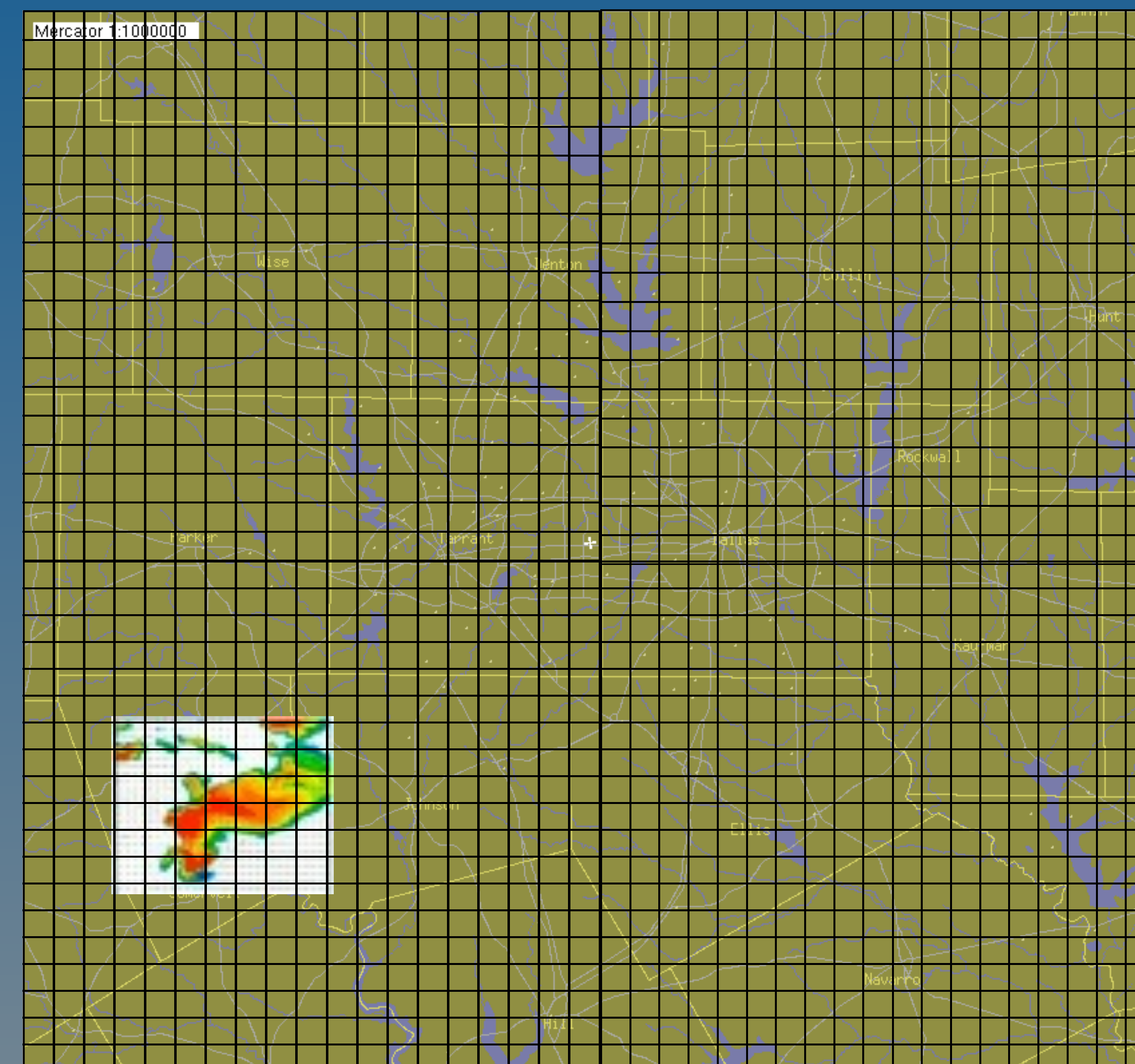
How far have we come? Resolving (sort of) a single storm!

1975



LFM Grid Point ($\Delta x \sim 190$ km)
7 vertical levels

2005

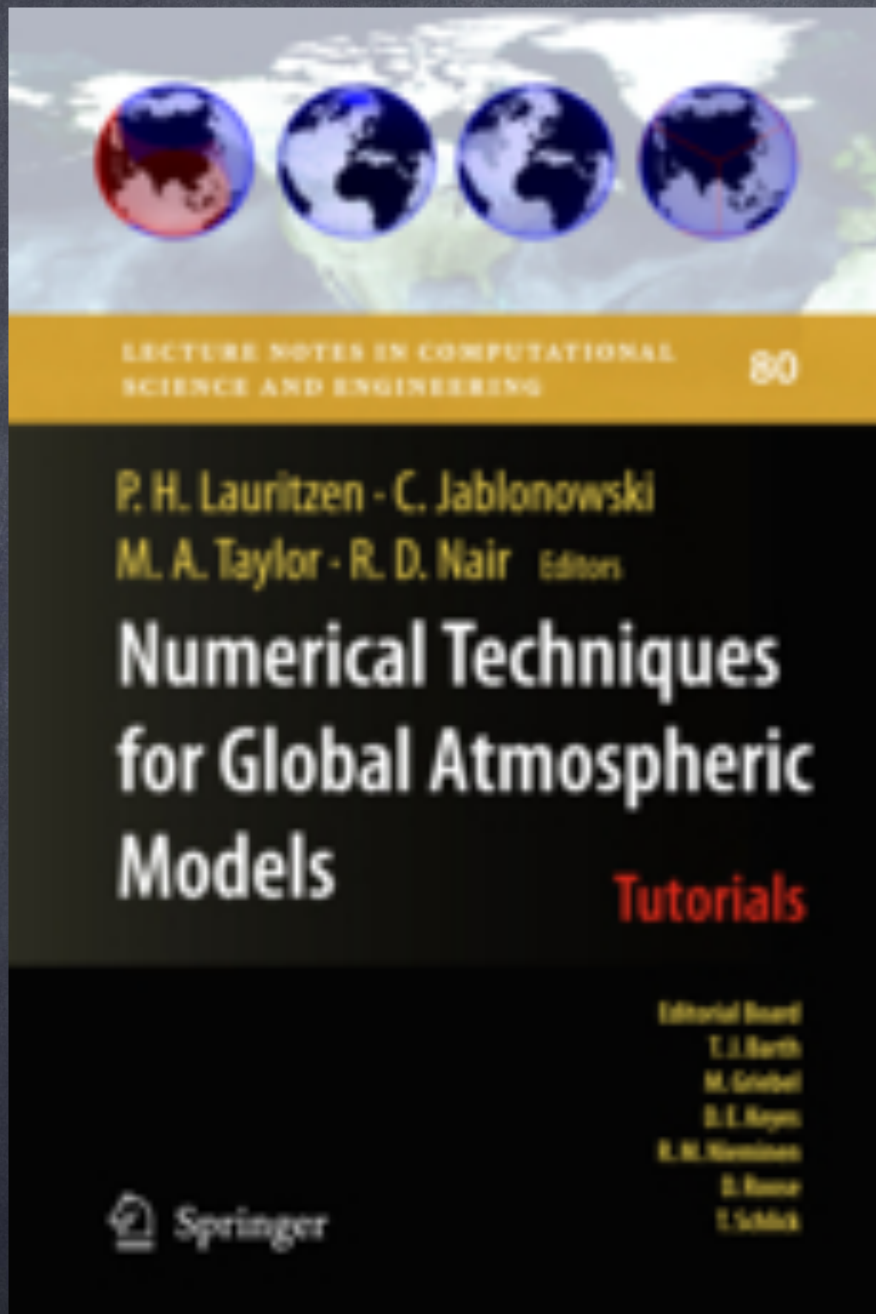


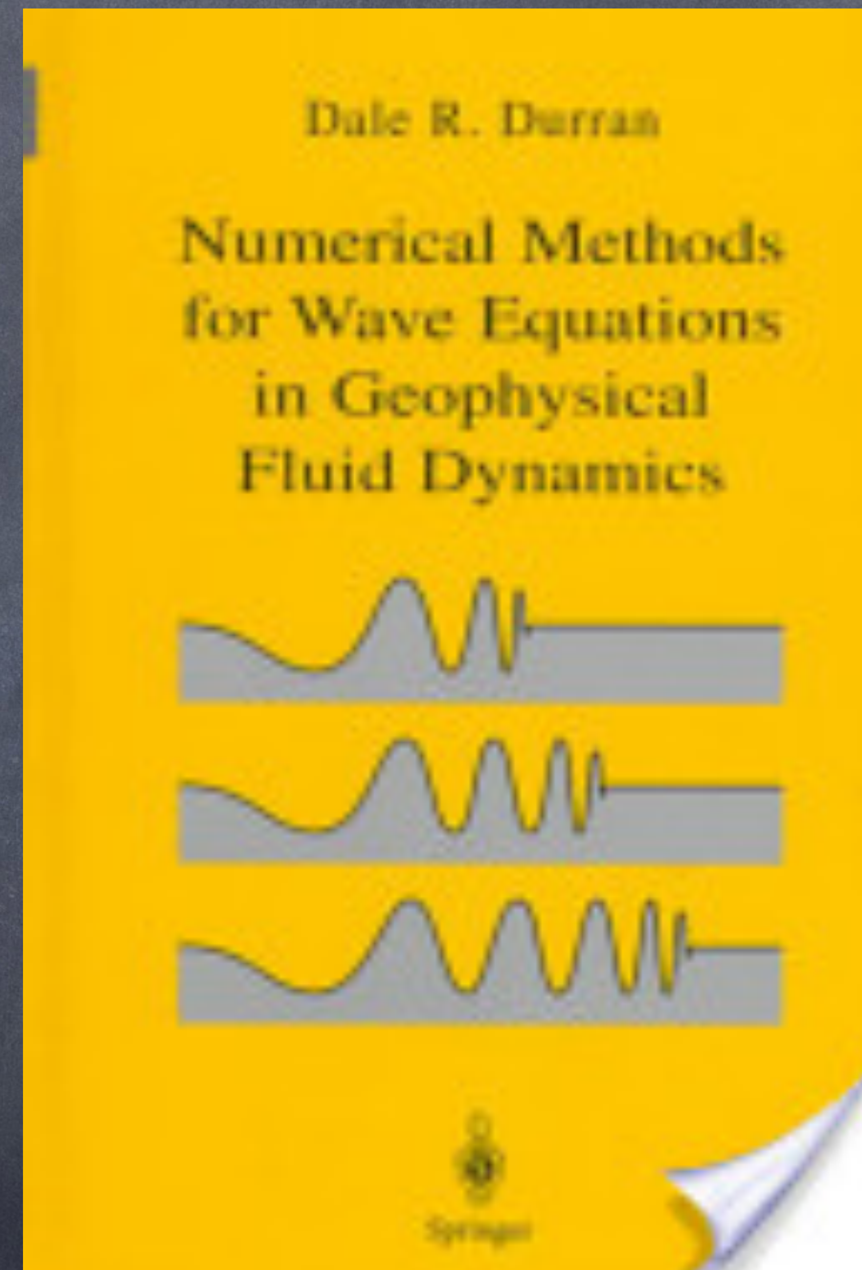
WRF Grid ($\Delta x \sim 4$ km)
50 vertical levels

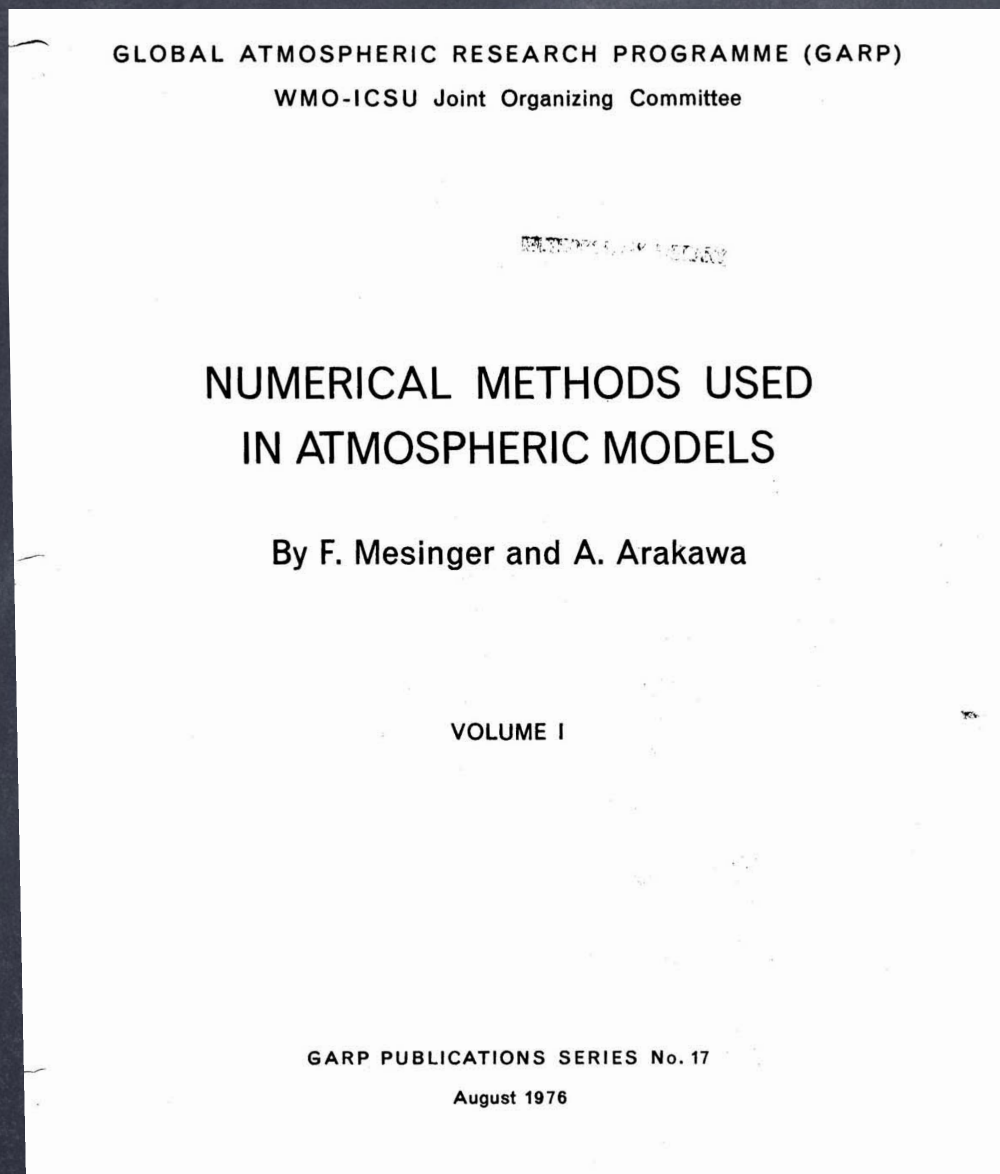
A ~35,000x increase in CPU due to grid! (really more like ~10^6 increase with physics changes)
A typical forecast today (1 hour wallclock) would require > 5 years to run on a 1975 computer!

Outline

- Introduction
- Equation sets
- Horizontal and Vertical Grids
- Numerical Methods
- Parameterizations in 50 min


LECTURE NOTES IN COMPUTATIONAL SCIENCE AND ENGINEERING
80
P. H. Lauritzen · C. Jablonowski
M. A. Taylor · R. D. Nair Editors
Numerical Techniques for Global Atmospheric Models
Tutorials
Springer


Dale R. Durran
Numerical Methods for Wave Equations in Geophysical Fluid Dynamics


GLOBAL ATMOSPHERIC RESEARCH PROGRAMME (GARP)
WMO-ICSU Joint Organizing Committee

NUMERICAL METHODS USED IN ATMOSPHERIC MODELS

By F. Mesinger and A. Arakawa

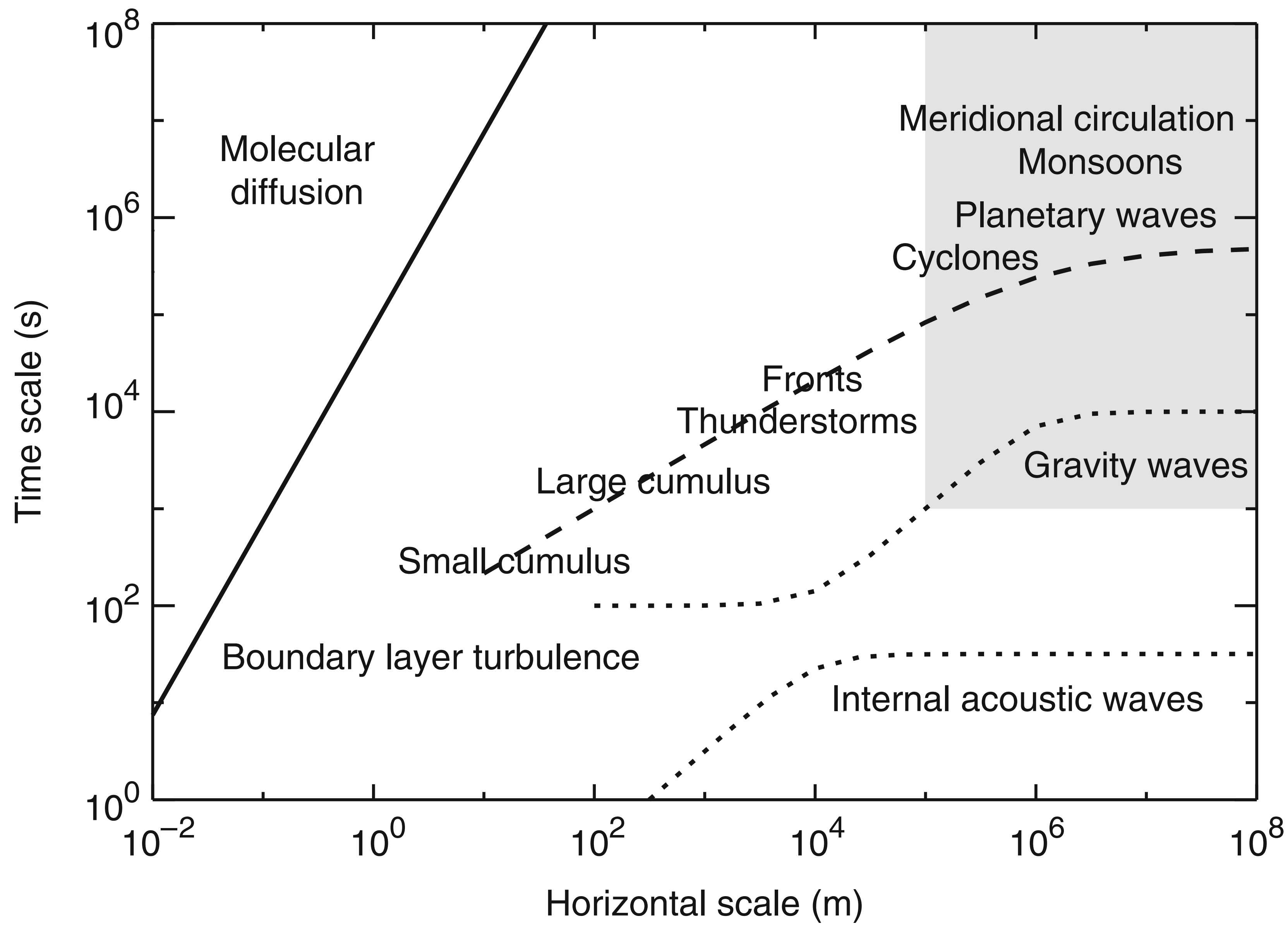
VOLUME I

GARP PUBLICATIONS SERIES No. 17
August 1976

Thomas Warner (NCAR)
NWP notes

Begin Lecture 1

Multiscale Nature of Atmospheric Dynamics



What is a Computer Model?

- Take the equations of fluid mechanics and thermodynamics that describe atmospheric processes.
- Convert them to a form where they can be programmed into a large computer.
- Solve them so that this software representation of the atmosphere evolves within the computer.
- This is called a “model” of the atmosphere

Scales of Models

Area Coverage and Resolution

- Global models - span the planet, represent large-scale atmospheric processes (dx ~ 20 km)
- Limited-area synoptic scale and mesoscale models – span continental, to state, to metro-areas; represent smaller-scale atmospheric processes (dx ~ 3 km)
- Computational fluid-dynamics (CFD) models – resolve flow around buildings, in street canyons, aircraft, etc. (dx ~ 1–100 m)

Scales of processes vs models

Global

- Planetary waves
- El Nino
- MJO
- AO

Synoptic

- Jet streams
- High and low pressure centers
- Troughs and ridges
- Fronts

Meso

- Thunderstorms
- Convective complexes
- Tropical storms
- Land/sea breezes
- Mountain/valley breezes
- Downslope wind storms
- Gap flows
- Cold air damming
- Nocturnal low-level jets
- Lake-effect snow bands

Urban

- Street-canyon flows
- Channeling around buildings, wakes
- Vertical transport on upwind and warm faces of buildings
- Flow in subway tunnels

Basic Equations

- Apply to many different types of atmospheric models
  - operational weather prediction models
  - global climate models
  - building-scale urban (CFD) models
  - research atmospheric models
  - models of flow over an airfoil
- In all cases, they are the equations of fluid dynamics applied to the atmosphere

Governing Equations

- Conservation of momentum (Newton's 2'd law)
  - 3 equations for accelerations of 3D winds (F = Ma)
- Conservation of mass
  - 1 equation for conservation of air (mass continuity)
  - 1 equation for conservation of water
- Conservation of energy
  - 1 equation for the first law of thermodynamics
- Relationship between p, V, and T
  - 1 equation of state (ideal gas law)

More on equations.....

- Almost every model uses a slightly different set of equations.

- Why?
  - –Application to different parts of the world
  - –Focus on different atmospheric processes
  - –Application to different time and spatial scales
  - –Ambiguity and uncertainty in formulations
  - –Tailoring to different uses

Starting Pt: Euler Equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$ cons. of mass

Diabatic Heating

$$\frac{D\theta}{Dt} = Q,$$ cons. of energy

$$\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla \Phi + \mathbf{F}.$$ momen. eq (3D)

Grav potential

$$\theta = T\left(\frac{p_0}{p}\right)^{\kappa}$$ Eq. of State

Solving an equation example.....

The equations describe how the atmosphere changes with time. For example, one equation would be:

$$\frac{\partial Temp}{\partial t} = -u \cdot \nabla Temp + \dot{Q}_{short} + \dot{Q}_{long}$$

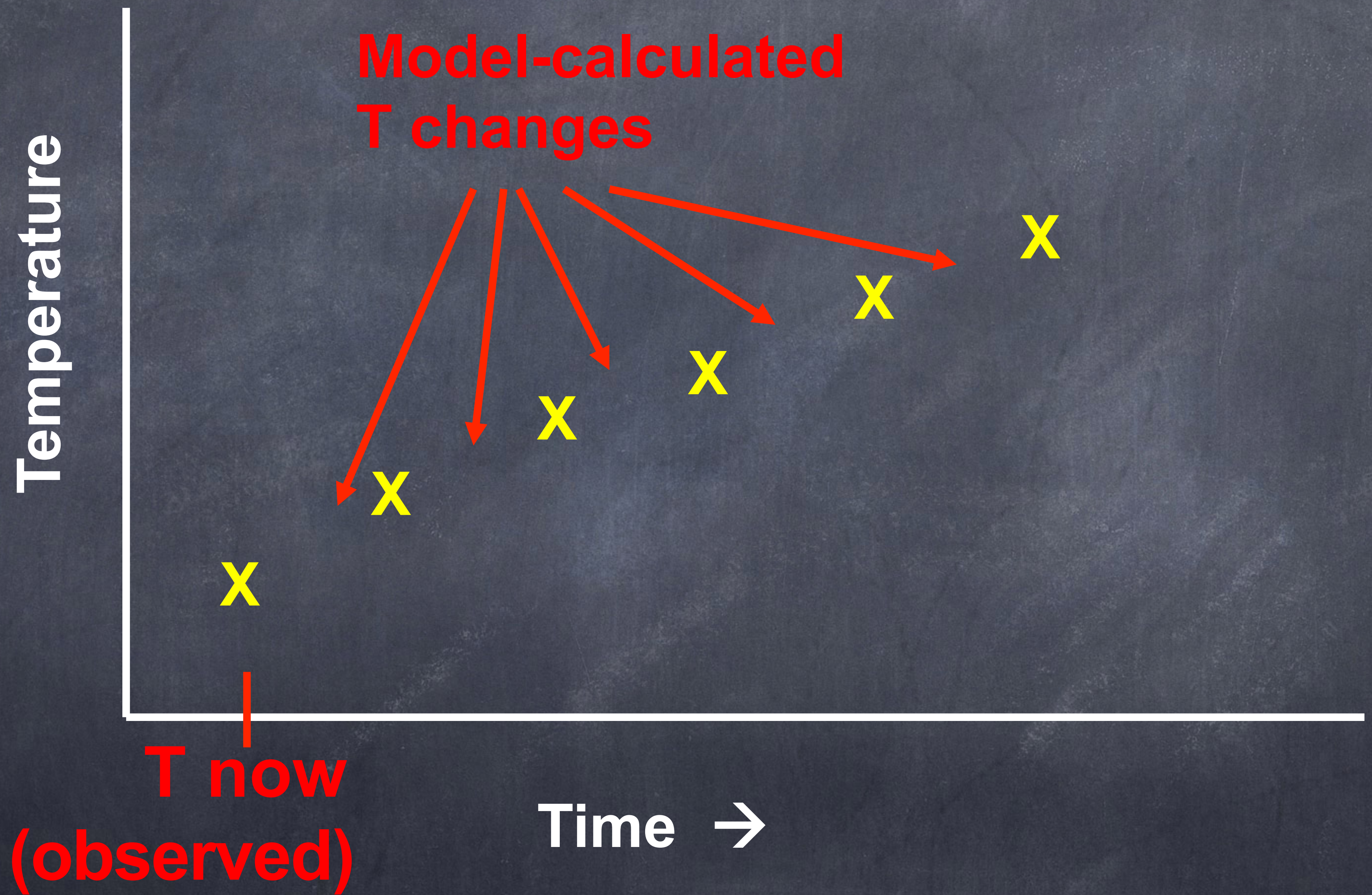
$$+\dot{Q}_{conduction} + \dot{Q}_{convection} + \dot{Q}_{longwave} + \dot{Q}_{evap} + \dot{Q}_{cond}$$

Solving an equation example.....

The equations describe how the atmosphere changes with time. For example, one equation would be:

Change in Temp at a point = advection
\+ shortwave radiation +longwave radiation
\+ conduction + convection
\+ evaporation + condensation

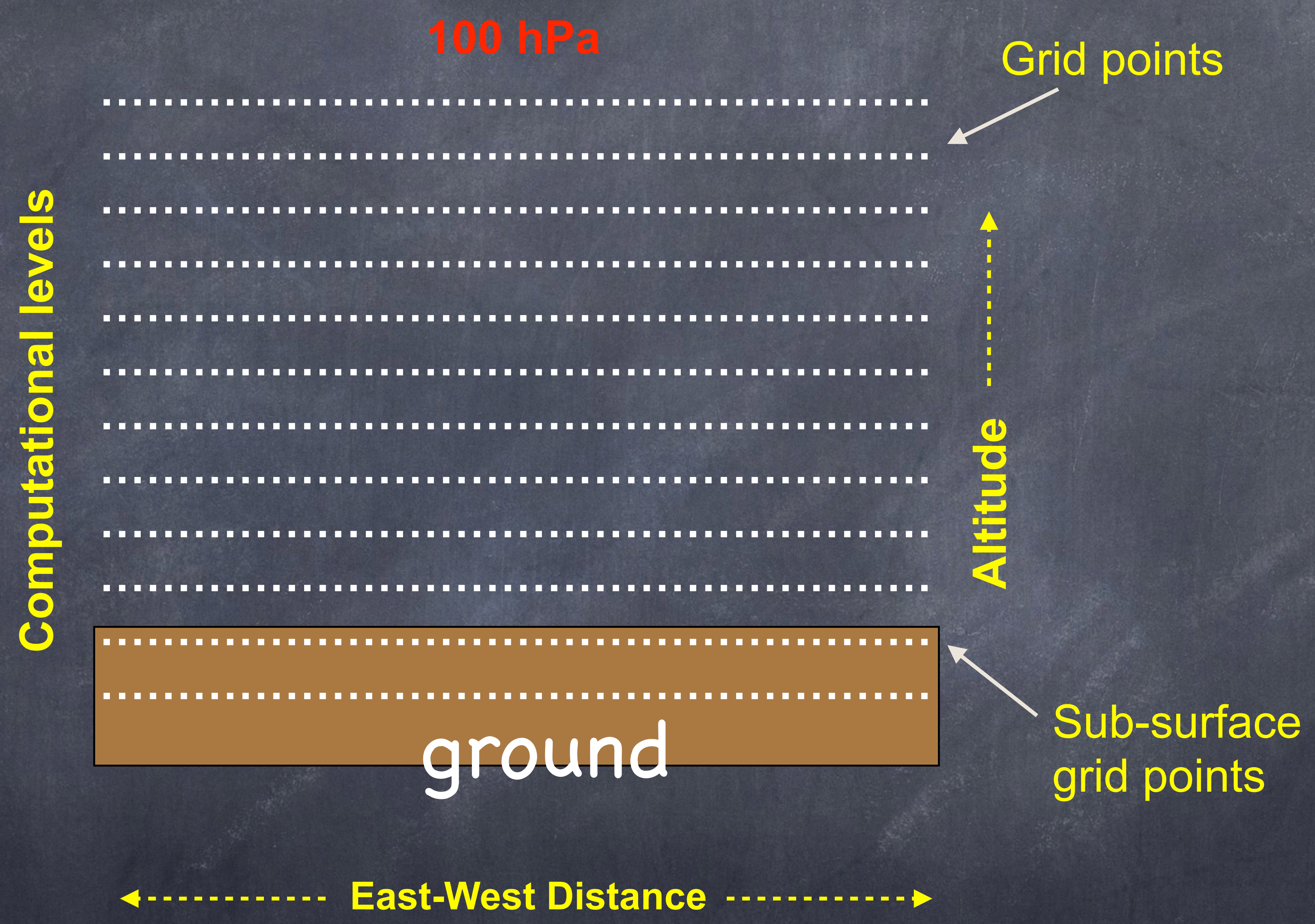
How the Model Forecasts



How a Model "Forecasts"

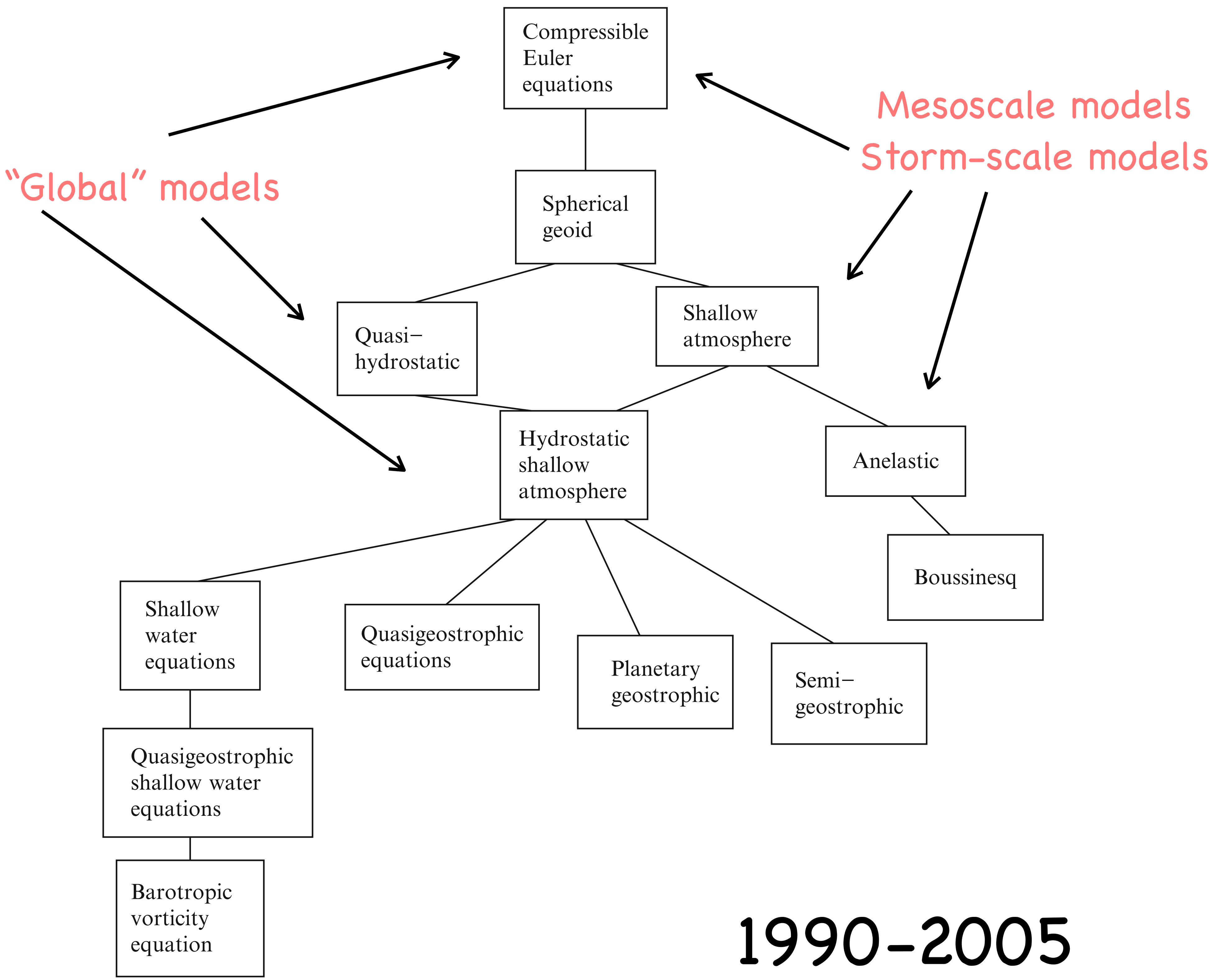
- This equation is solved for a three-dimensional "matrix" of points (or a grid) that covers the atmosphere from the surface to some level near the top of the atmosphere.
- Here is a 2-dimensional slice through the grid in the X-Z plane (west-2-east, sfc-2-trop)

2D Model Grid

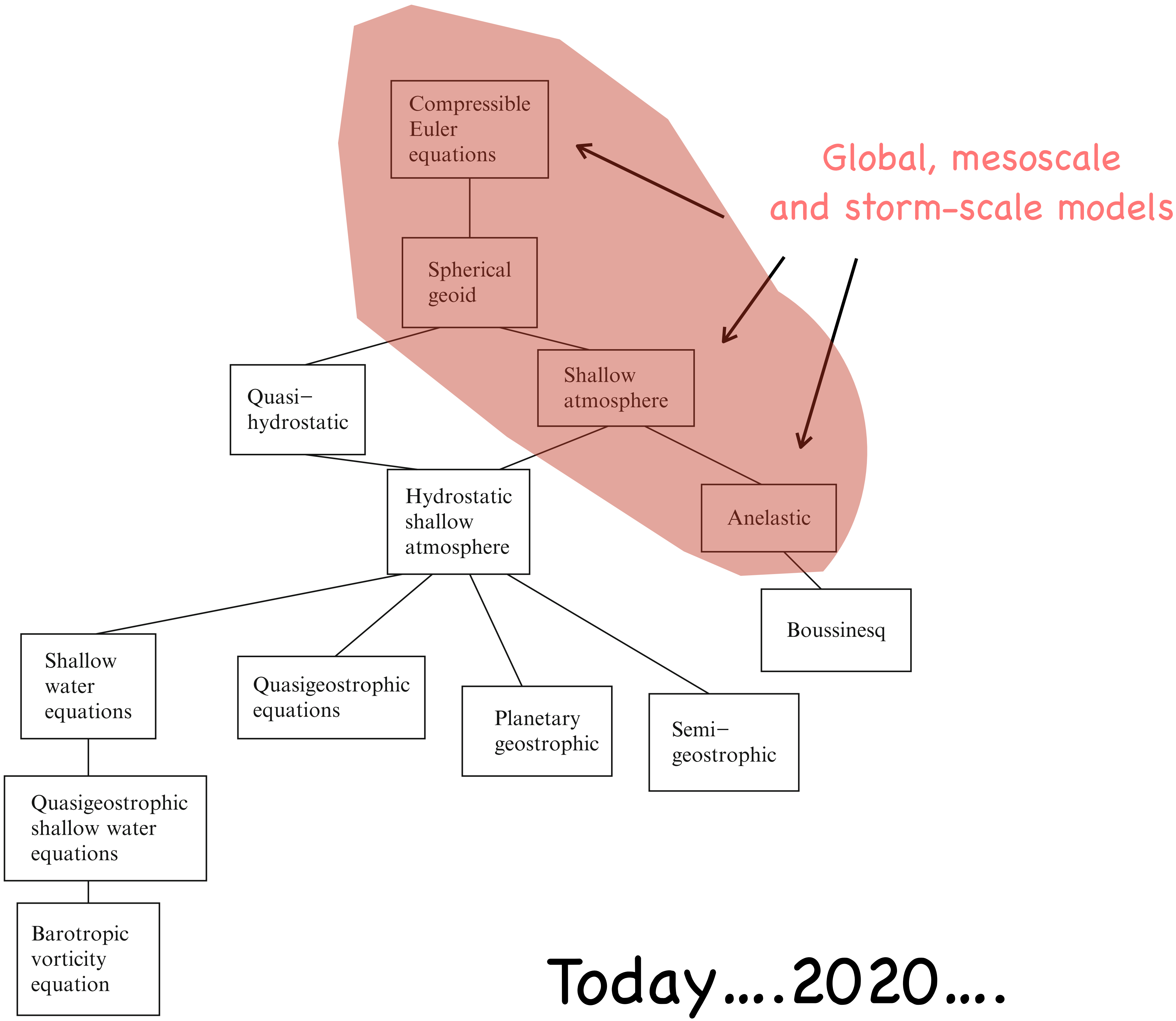


Approximate Eq. Sets

- Spherical geoid: assume the earth is a perfect sphere, so gravity is parallel to radius direction from earth's center. Often used.
  - fastest wave speed: U(100) + Sound waves (350) ~ 450 m/s
- (Quasi) Hydrostatic approximation: Dw/Dt=0 Good for horizontal scales > 10 km.
  - fastest wave speed: U(100) + Lamb (surface) wave (350) ~ 450 m/s
- Anelastic approximation: form of incompressibility - filters out sound waves from system. Good approximation for meso- and storm-scale (maybe not planetary scales).
  - fastest wave speed: U(100) + Gravity waves (50) ~ 150 m/s
- Shallow atmosphere:
  - assume 1/r in equations ~ 1/a (neglect distance above ground)
  - Coriolis acts only in vertical
  - a few other small terms associated with spherical metrics neglected


Compressible Euler equations
Spherical geoid
Quasi–hydrostatic
Shallow atmosphere
Hydrostatic shallow atmosphere
Anelastic
Boussinesq
Shallow water equations
Quasigeostrophic equations
Planetary geostrophic
Semi–geostrophic
Quasigeostrophic shallow water equations
Barotropic vorticity equation
"Global" models
Mesoscale models
Storm-scale models

1990-2005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Barotropic vorticity equation
Global, mesoscale and storm-scale models

Today….2020….

3D Equations

- Requirements? (George Bryan talk)
  - non-hydrostatic & compressible
  - minimum number of approximations
  - solver should conserve mass
- Energy Conservation?
  - small scales and short integrations: desired
  - global scale (climate): REQUIRED

3D Equations

- Why mass?
  - transport/dispersion applications
  - longer integration times
  - certain convective applications (hurricane) this makes a big difference
- Why not energy conservation?
  - Its hard! Complexity
  - Dissipative heating
    - sub-grid turbulence
    - higher-order diffusion
    - PBL parameterization
  - Moist processes
    - sedimentation of hydrometeors
    - dissipative heating around falling hydrometeors
    - no agreement about exact form of equations with mixtures of ice and water

Equation Set 1

(Giraldo and Restelli JCP 2009)

Equation set 1, which has been used extensively in mesoscale modeling, reads

$$
\begin{aligned}
&\frac{\partial \pi}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \pi + \frac{R}{c_v} \pi \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \\
&\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + c_p \theta \boldsymbol{\nabla} \pi = -g\boldsymbol{k} + \mu \nabla^2 \boldsymbol{u}, \\
&\frac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \theta = \mu \nabla^2 \theta,
\end{aligned}
\tag{1}
$$

where the solution vector is $(\pi, \boldsymbol{u}^{\mathcal{T}}, \theta)^{\mathcal{T}}$, $\pi = \left(\frac{P}{P_0}\right)^{R/c_p}$ is the Exner pressure, $\boldsymbol{u} = (u, w)^{\mathcal{T}}$ is the velocity field, $\theta = \frac{T}{\pi}$ is the potential temperature, and $\mathcal{T}$ denotes the transpose operator. In these equations P is the pressure,

a.ka., Klemp-Wilhelmson/MM5 equations

(K&W JAS, 1978)

Equation Set 2

(Giraldo and Restelli JCP 2009)

Equation set 2 is gaining popularity in the literature because it is not too dissimilar from set 1 and is in conservation form (for the inviscid case only). These equations are written as follows:

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0, \\
&\frac{\partial \rho \boldsymbol{u}}{\partial t} + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u} + P \mathcal{I}_2) = -\rho g \boldsymbol{k} + \nabla \cdot (\mu \rho \nabla \boldsymbol{u}), \qquad (4) \\
&\frac{\partial \rho \theta}{\partial t} + \nabla \cdot (\rho \theta \boldsymbol{u}) = \nabla \cdot (\mu \rho \nabla \theta),
\end{aligned}$$

where the conserved variables are $(\rho, \rho \boldsymbol{u}^{\mathcal{T}}, \rho\theta)^{\mathcal{T}}$, ρ is the density, $\boldsymbol{u} = (u, w)^{\mathcal{T}}$ is the velocity field, and θ is the potential temperature which we have defined previously. The pressure P which appears in the momentum equation is obtained by the equation of state

$$P = P_0 \left(\frac{\rho R \theta}{P_0} \right)^{\frac{c_p}{c_v}} \qquad (5)$$

a.ka., WRF/MPASS equations

(Klemp et al. 2007 MWR)

Equation Set 3

(Giraldo and Restelli JCP 2009)

These equations are written as follows:

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0, \\
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&\frac{\partial \rho e}{\partial t} + \nabla \cdot [(\rho e + P)\boldsymbol{u}] = \nabla \cdot \boldsymbol{F}_{e}^{\text{visc}},
\end{aligned} \tag{7}$$

where the conserved variables are $(\rho, \rho \boldsymbol{u}^{\mathcal{T}}, \rho e)^{\mathcal{T}}$, ρ is the density, $\boldsymbol{u} = (u, w)^{\mathcal{T}}$ is the velocity field, $e = c_v T + \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u} + \varphi$ is the total energy, and $\varphi = gz$ is the geopotential height. The pressure P is obtained by the equation of state which, in terms of the solution variables, is written as

$$P = \frac{R}{c_v}\rho\left(e - \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u} - \varphi\right). \tag{8}$$

CFD equations

Total energy is thermodynamic variable

Equation Set 3

(Giraldo and Restelli JCP 2009)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0,$$

$$\frac{\partial \rho \boldsymbol{u}}{\partial t} + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u} + P\mathcal{I}_2) = -\rho g \boldsymbol{k} + \nabla \cdot \boldsymbol{F}_{\boldsymbol{u}}^{\text{visc}},$$

$$\frac{\partial \rho e}{\partial t} + \nabla \cdot [(\rho e + P)\boldsymbol{u}] = \nabla \cdot \boldsymbol{F}_{e}^{\text{visc}},$$

The viscous fluxes $\boldsymbol{F}^{\text{visc}}$ are defined as follows:

$$\boldsymbol{F}_{\boldsymbol{u}}^{\text{visc}} = \mu[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathcal{T}} + \lambda(\nabla \cdot \boldsymbol{u})\mathcal{I}_2]$$

and

$$\boldsymbol{F}_{e}^{\text{visc}} = \boldsymbol{u} \cdot \boldsymbol{F}_{\boldsymbol{u}}^{\text{visc}} + \frac{\mu c_p}{Pr} \nabla T,$$

Equation Set 3

(Giraldo and Restelli JCP 2009)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0,$$
$$\frac{\partial \rho \boldsymbol{u}}{\partial t} + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u} + P\mathcal{I}_2) = -\rho g \boldsymbol{k} + \nabla \cdot \boldsymbol{F}_{\boldsymbol{u}}^{\text{visc}},$$
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and

$$\boldsymbol{F}_e^{\text{visc}} = \boldsymbol{u} \cdot \boldsymbol{F}_{\boldsymbol{u}}^{\text{visc}} + \frac{\mu c_p}{Pr} \nabla T,$$

Viscous heating!
Dissipation adds heat to system!
Important in hurricanes BL!

Most atmospheric models DO NOT account for this (especially cloud models!)

NWP Basics: Important Choices...

- **What are you trying to predict?**
  - T/Wind/Visibility/Rainfall/Precip Type
  - Severe storms? Hurricanes?
- What resolution do you need to predict it?
- How accurate is your model predicting it?
- How fast can it run on your computer?
- (When do your users need the output?)

What's important?

- **What are you trying to predict?**
- What resolution do you need to predict it?
- How fast can it run on your computer?
- How accurate is your model predicting it?

Factors controlling model efficiency....

- Length of forecast
- Size of numerical grid (resolution needed)
- Time step needed for <u>stable</u> integration
- Time step needed for <u>accurate</u> integration
- Cost of dynamical core
- Cost of physics (usually >> dynamical core)
- How well model "scales" on your operational computer

End Lecture 1

Begin Lecture 2

Notes are available at
http://www.nssl.noaa.gov/users/lwicker/public_html/NWP_5004.pdf

Solution Basics

An example of one momentum equation:
- 1-d wind accelerated by only the pressure gradient force

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

Computers cannot analytically solve even this very simple equation!
Why?

Solution Basics

- The problem: computers can perform arithmetic but not calculus

$$+ \quad - \quad \times \quad \div$$

$$\frac{d(f)}{dx} \qquad \int (f)\,dx$$

- The solution: numerical methods

Solution Basics

1-D Advection of a parcel (analytical)

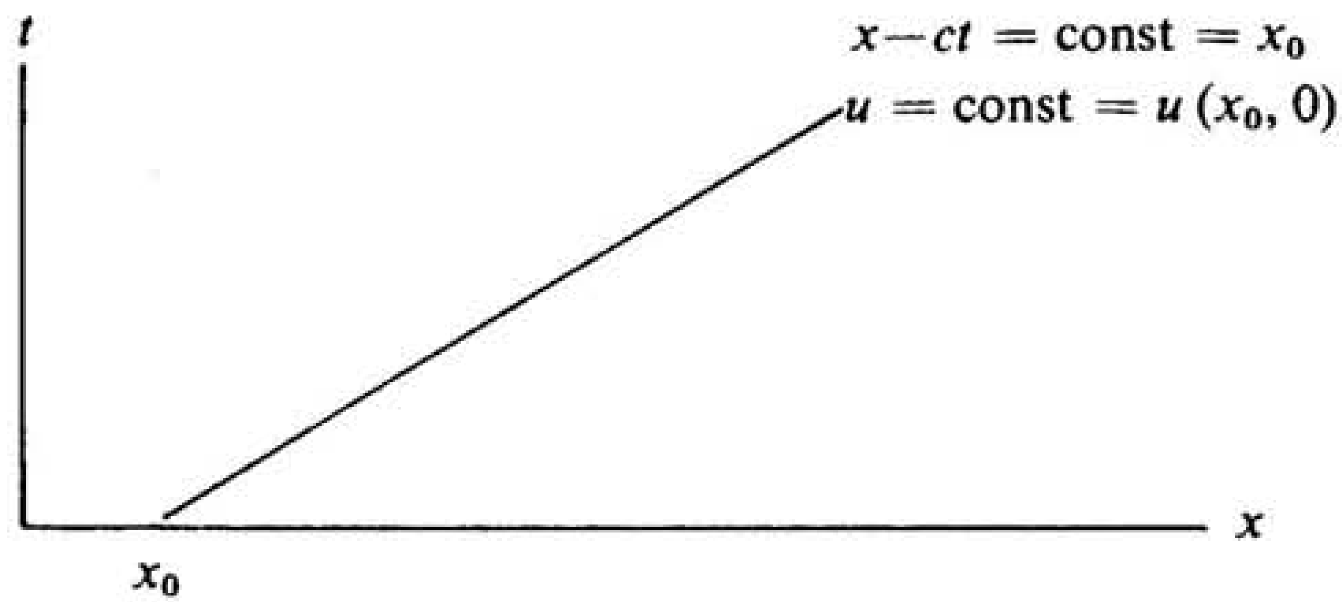


Figure 4.1 One of the characteristics of the linear advection equation (4.1).

U > 0

Solution Basics

The simplest model grid?

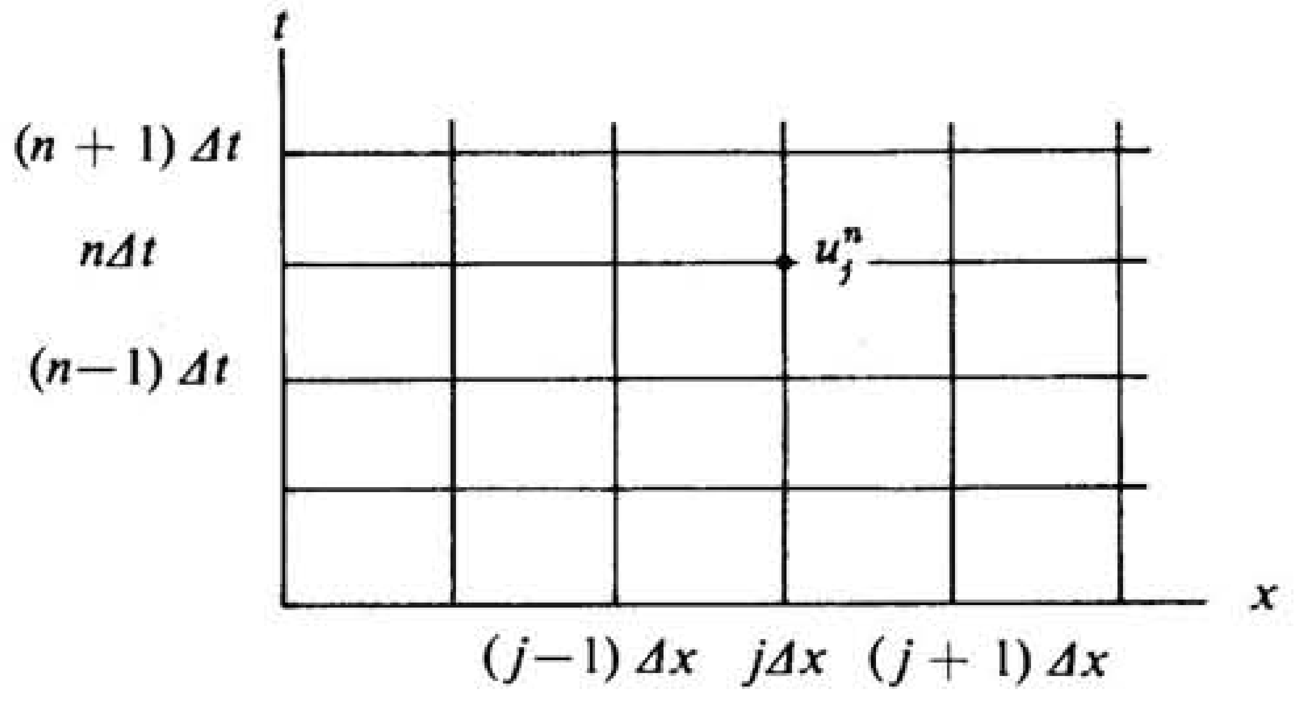


Figure 4.2 A finite difference grid for finding an approximate solution of (4.1).

1-D in space grid

1-D in time grid

Solution Basics

use knowledge of analytical PDE solution to approximate solution……

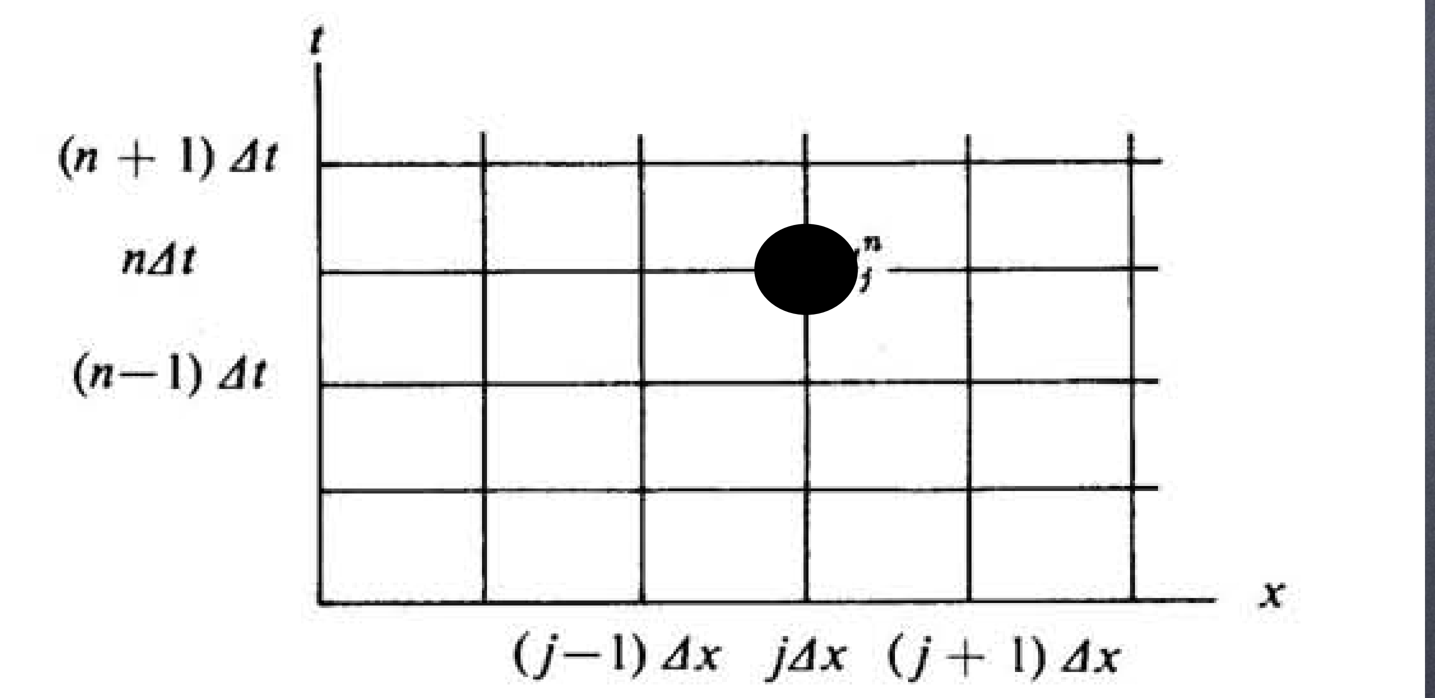


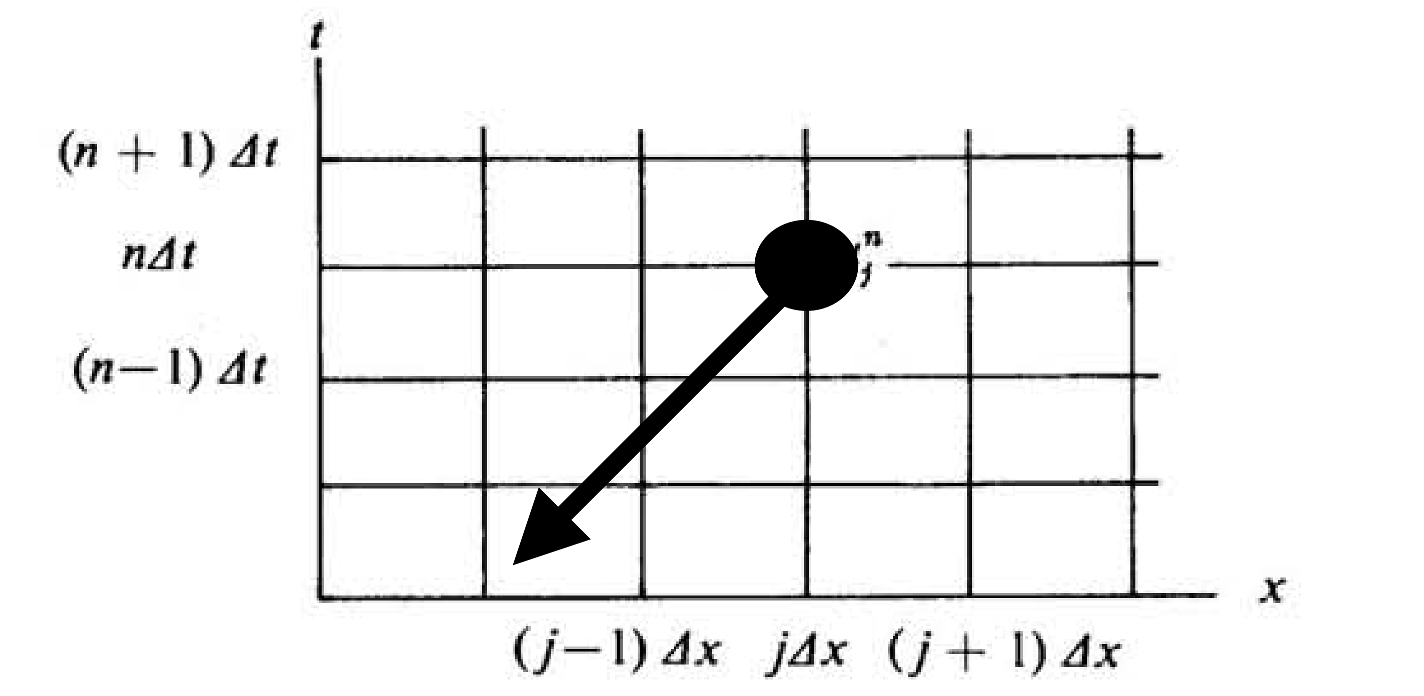
Figure 4.2 A finite difference grid for finding an approximate solution of (4.1).

U > 0

$$\frac{T_j^{n+1} - T_j^n}{\Delta t} = -U_o \frac{T_j^n - T_{j-1}^n}{\Delta t}$$

Solution Basics

use knowledge of analytical PDE solution to approximate solution……



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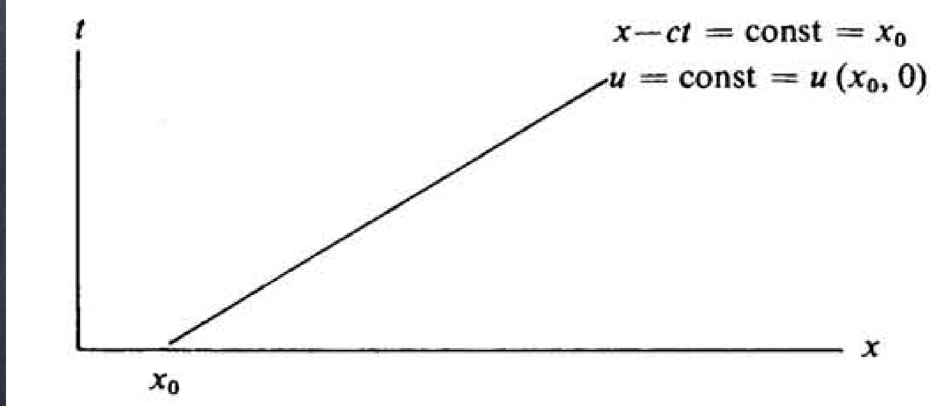

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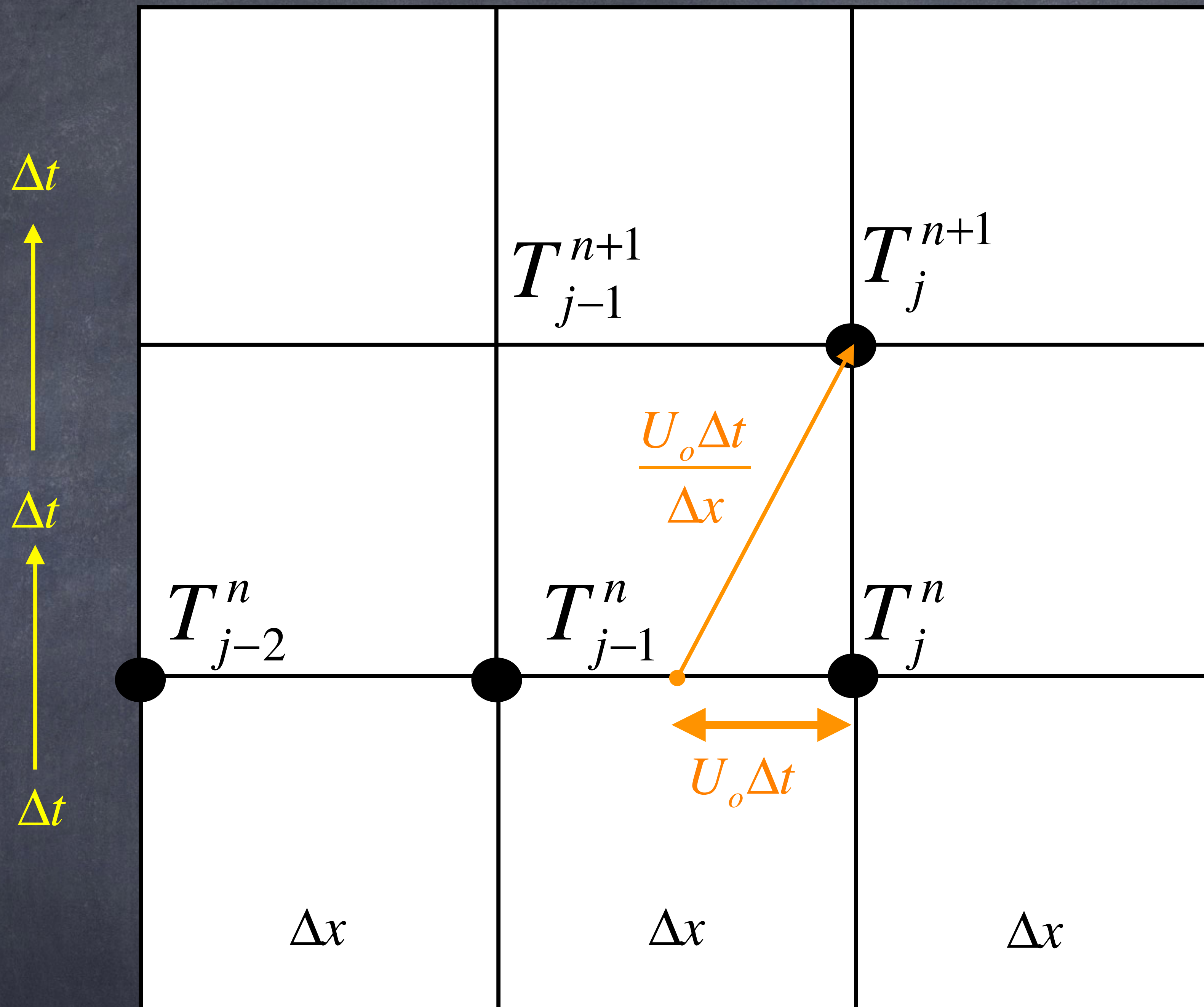
$$\frac{T_j^{n+1} - T_j^n}{\Delta t} = -U_o \frac{T_j^n - T_{j-1}^n}{\Delta t}$$

Example: Upwind scheme

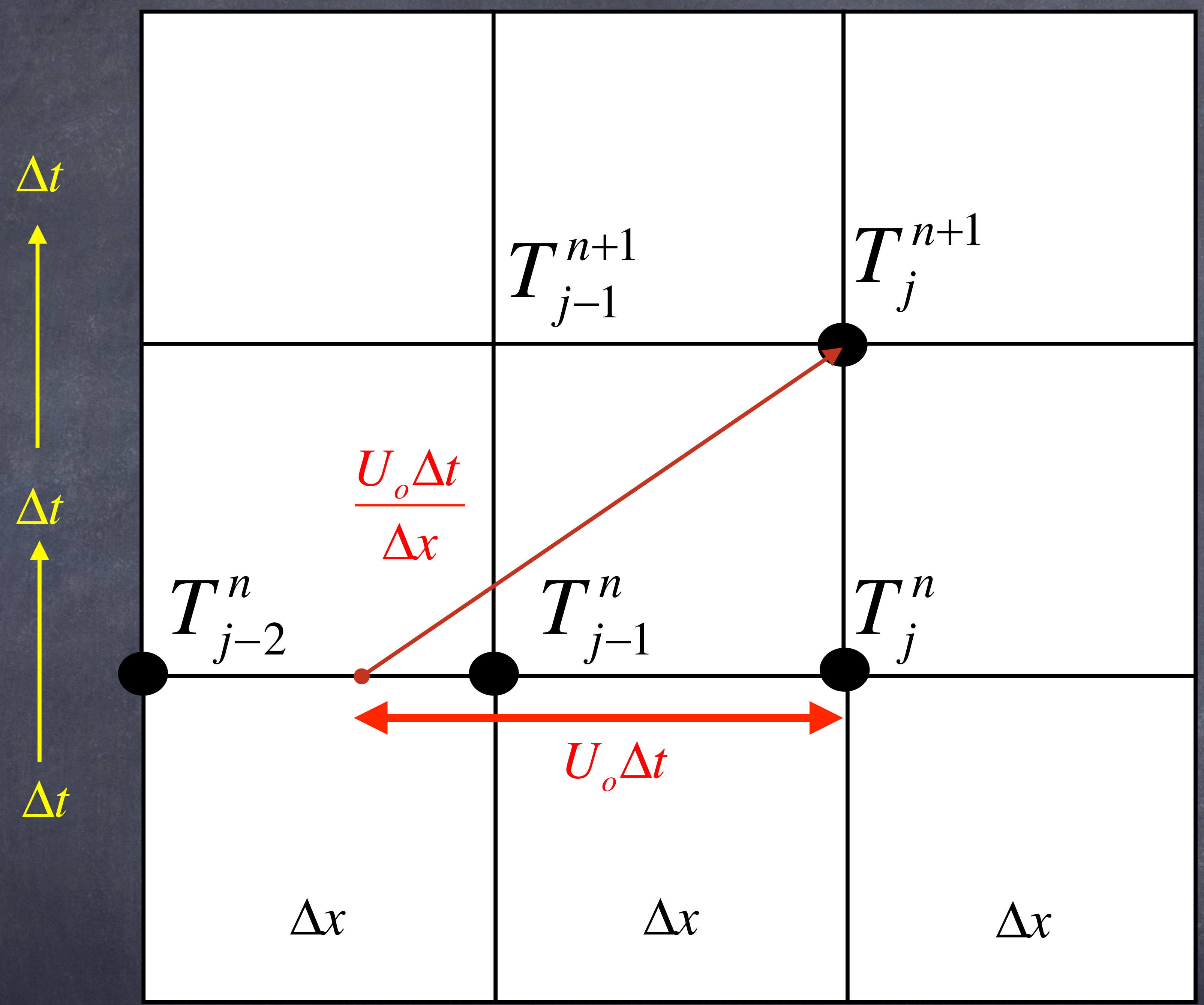


$$U_o\Delta t \le \Delta x$$

Stable!

$$\frac{T_j^{n+1}-T_j^n}{\Delta t} = -U_o\frac{T_j^n - T_{j-1}^n}{\Delta x} \quad \dashrightarrow \quad T_j^{n+1} = T_j^n - \frac{U_o\Delta t}{\Delta x}\left(T_j^n - T_{j-1}^n\right)$$

Example: Upwind scheme



$$U_1 \Delta t \geq \Delta x$$

UNSTABLE!

$$\frac{T_j^{n+1} - T_j^n}{\Delta t} = -U_1 \frac{T_j^n - T_{j-1}^n}{\Delta x} \quad \dashrightarrow \quad T_j^{n+1} = T_j^n - \frac{U_1 \Delta t}{\Delta x}\left(T_j^n - T_{j-1}^n\right)$$

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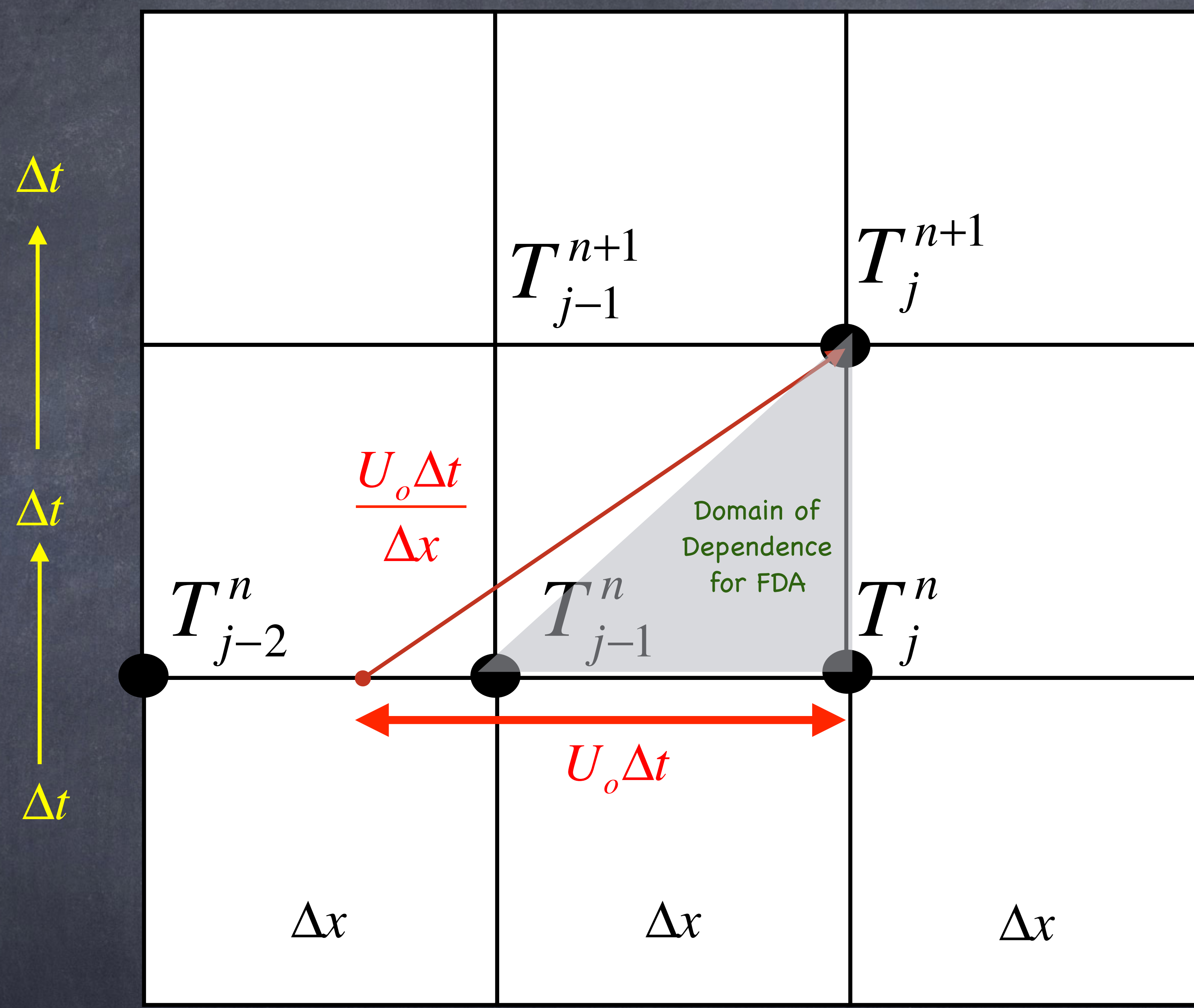

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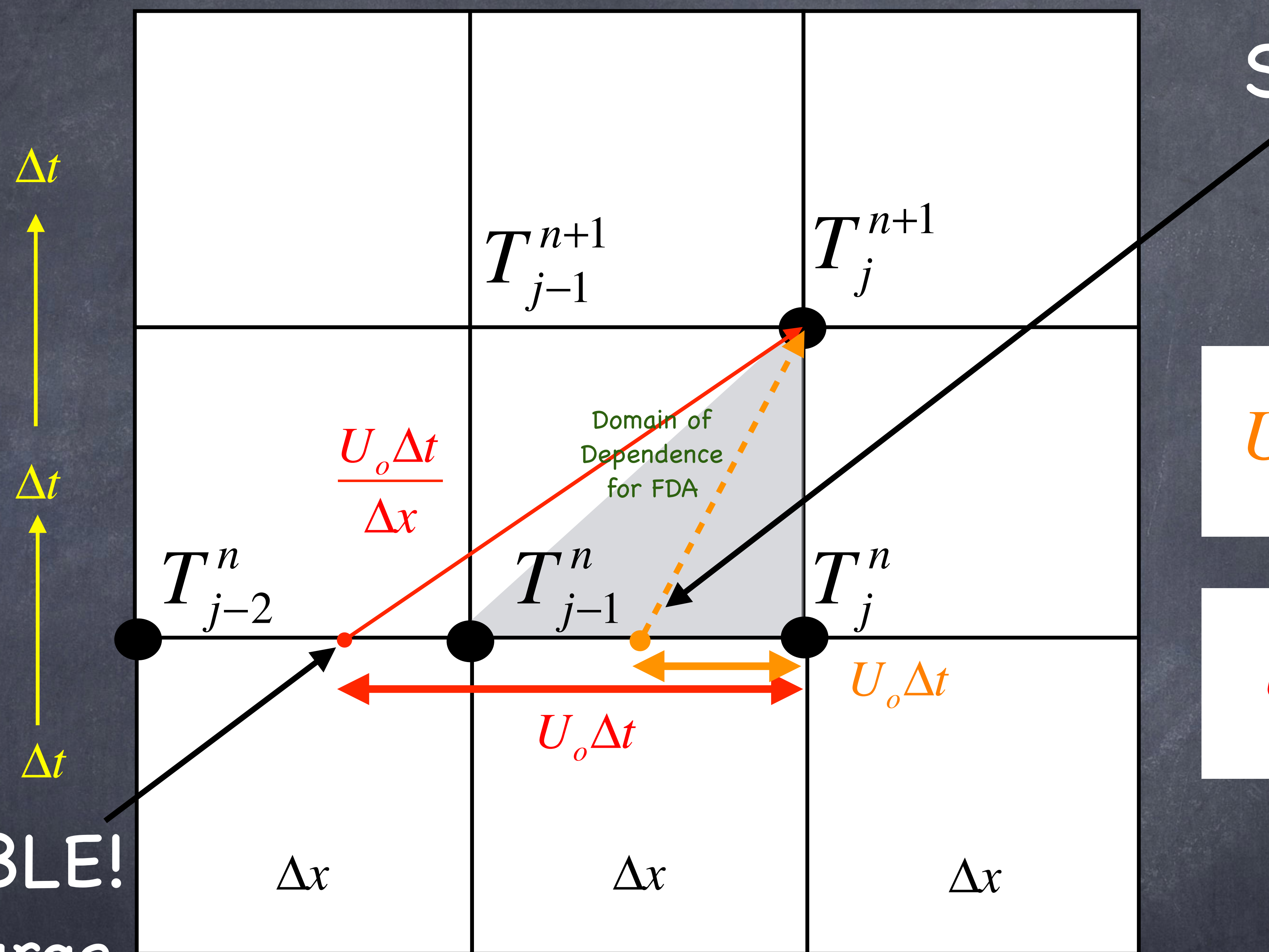
Example: Upwind scheme

STABLE!



$U_o \Delta t \leq \Delta x$

$U_1 \Delta t \geq \Delta x$

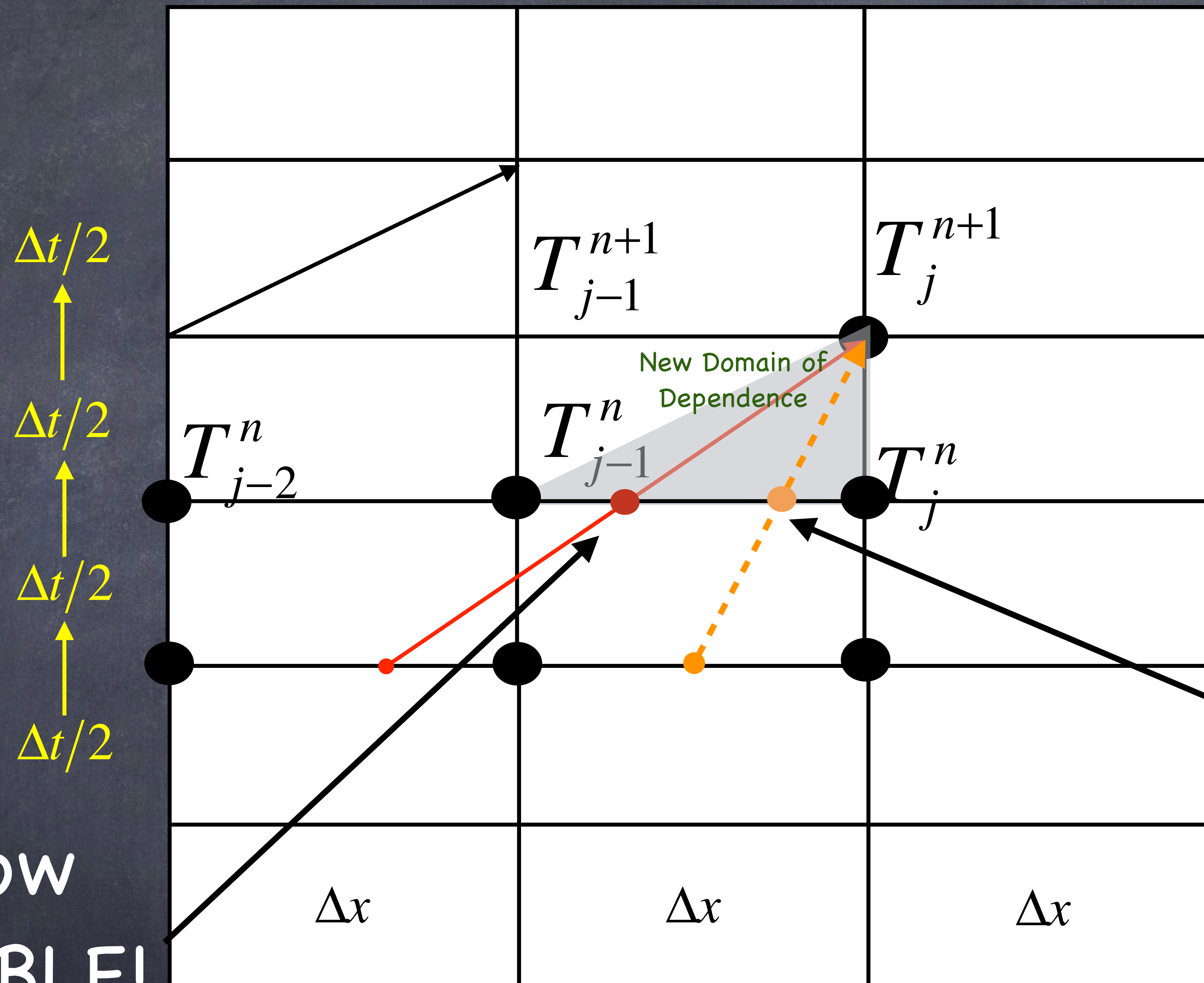
UNSTABLE!
dt too large

$$\frac{U \Delta t}{\Delta x} < 1$$

Needed for Stability!

CFL condition

Example: Upwind scheme



$$U_o \frac{\Delta t}{2} \leq \Delta x$$

$$U_1 \frac{\Delta t}{2} \leq \Delta x$$

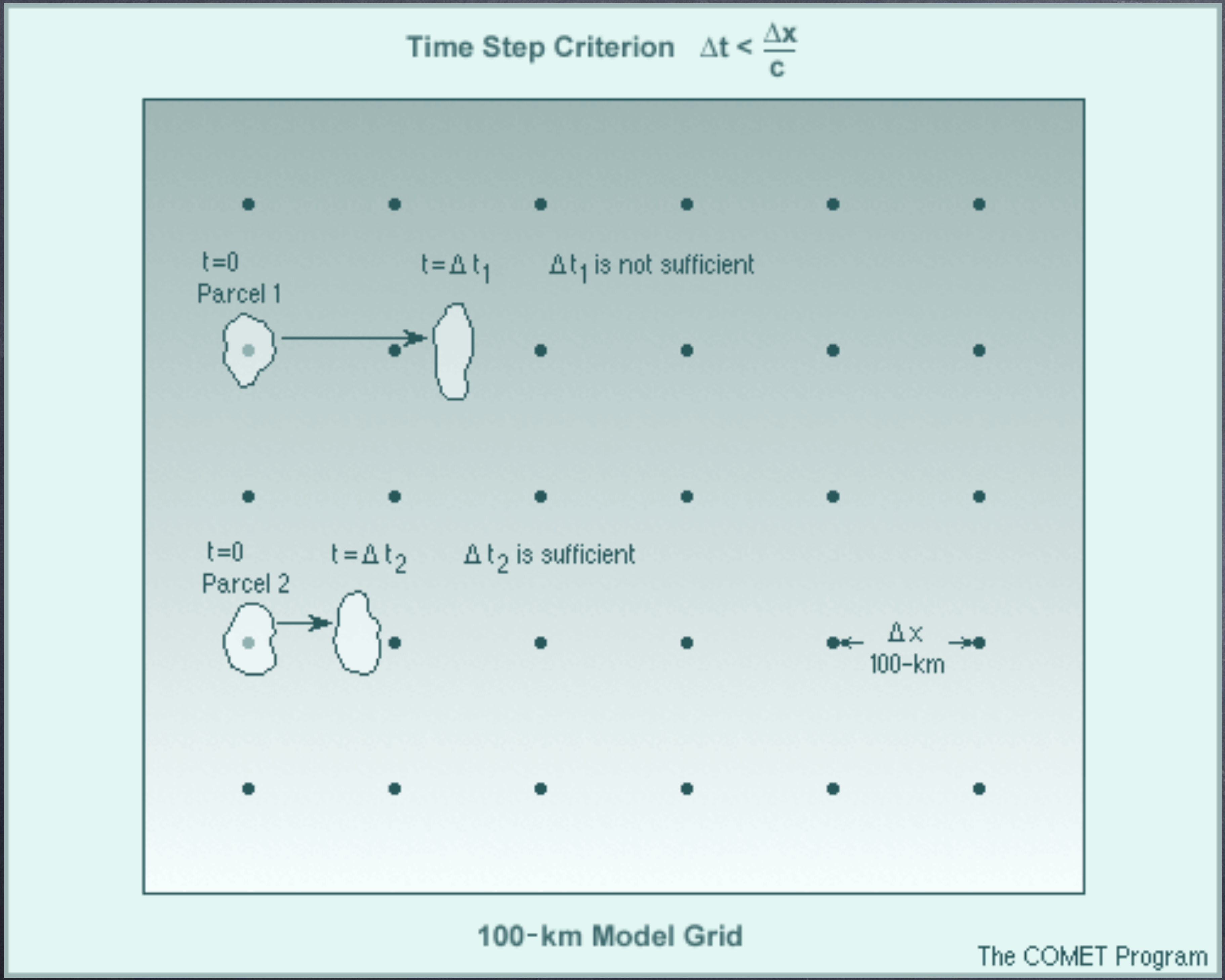
Still STABLE!

Now STABLE!

$$\frac{U \Delta t}{\Delta x} < 1$$

New time step does not violate CFL condition for either U

Solution Basics



<u>In NWP:</u>

dt is set to maintain CFL stability for a given dx and fastest flow/wave speeds

Approximate Eq. Sets

- Spherical geoid: assume the earth is a perfect sphere, so gravity is parallel to radius direction from earth's center. Often used.
  - fastest wave speed: U(100) + Sound waves (350) ~ 450 m/s
- (Quasi) Hydrostatic approximation: Dw/Dt=0 Good for horizontal scales > 10 km.
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- Shallow atmosphere:
  - assume 1/r in equations ~ 1/a (neglect distance above ground)
  - Coriolis acts only in vertical
  - a few other small terms associated with spherical metrics neglected

What do the PDEs look like?

Equations of motion (ECWMF model)

$$\frac{\partial U}{\partial t} + \frac{1}{a\cos^2\theta}\left\{U\frac{\partial U}{\partial \lambda} + v\cos\theta\frac{\partial U}{\partial \theta}\right\} + \dot{\eta}\frac{\partial U}{\partial \eta} + (-fv) + \frac{1}{a}\left\{\frac{\partial \phi}{\partial \lambda} + R_{\mathrm{dry}}T_{\mathrm{v}}\frac{\partial}{\partial \lambda}(\ln p)\right\} = \underline{P_U + K_U}$$

East-west wind

$$\frac{\partial V}{\partial t} + \frac{1}{a\cos^2\theta}\left\{U\frac{\partial V}{\partial \lambda} + V\cos\theta\frac{\partial V}{\partial \theta} + \sin\theta(U^2 + V^2)\right\} + \dot{\eta}\frac{\partial V}{\partial \eta} + fU + \frac{\cos\theta}{a}\left\{\frac{\partial \phi}{\partial \theta} + R_{\mathrm{dry}}T_{\mathrm{v}}\frac{\partial}{\partial \theta}(\ln p)\right\} = \underline{P_V + K_V}$$

North-south wind

$$\frac{\partial T}{\partial t} + \frac{1}{a\cos^2\theta}\left\{U\frac{\partial T}{\partial \theta} + V\cos\theta\frac{\partial T}{\partial \theta}\right\} + \dot{\eta}\frac{\partial T}{\partial \eta} - \frac{\kappa T_{\mathrm{v}}\omega}{(1 + (\delta - 1)q)p} = \underline{P_T + K_T}$$

Temperature

$$\frac{\partial q}{\partial t} = \frac{1}{a\cos^2\theta}\left\{U\frac{\partial q}{\partial \lambda} + V\cos\theta\frac{\partial q}{\partial \theta}\right\} = \eta\frac{\partial q}{\partial \eta} = \underline{P_q + K_q}$$

Humidity

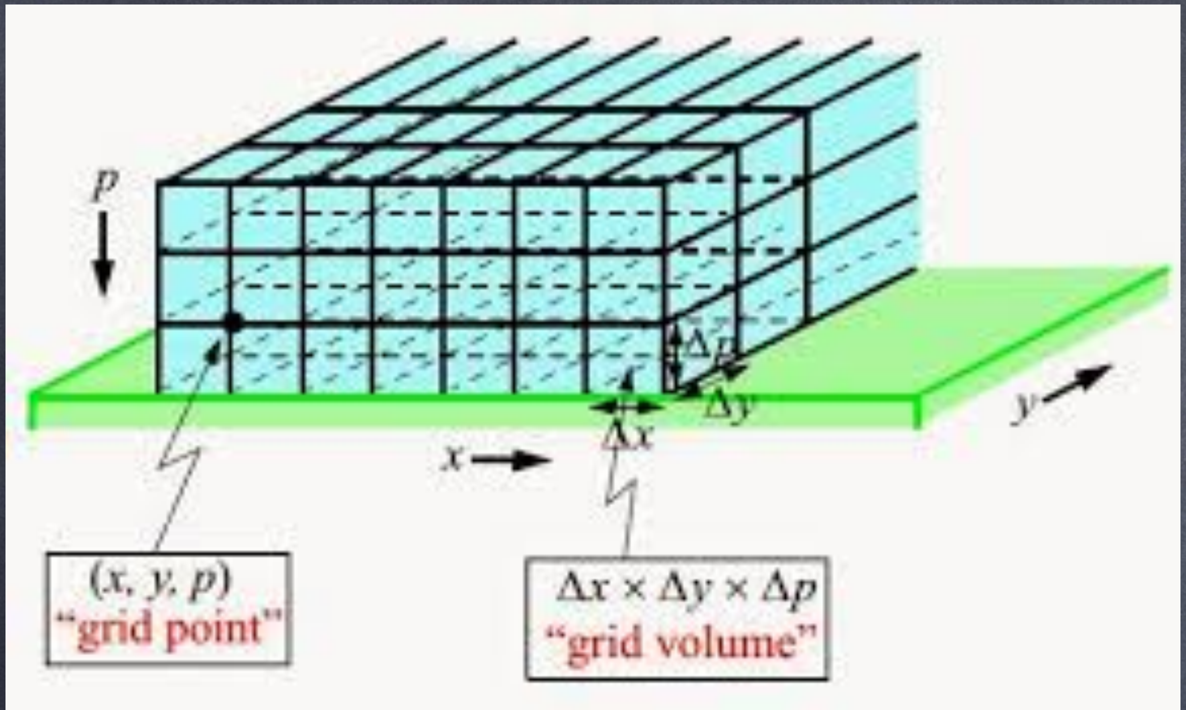
$$\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial \eta}\right) + \nabla\cdot\left(\mathbf{v}_{\mathrm{H}}\frac{\partial p}{\partial \eta}\right) + \frac{\partial}{\partial \eta}\left(\dot{\eta}\frac{\partial p}{\partial \eta}\right) = 0$$

Continuity of mass

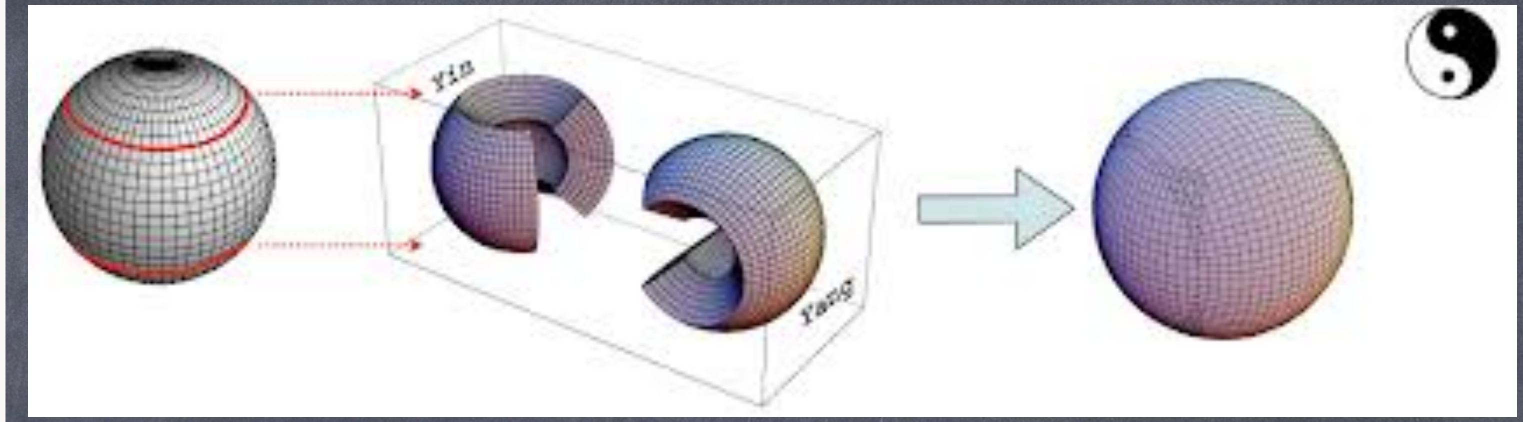
$$\frac{\partial p_{\mathrm{surf}}}{\partial t} = -\int_0^1 \nabla\cdot\left(\mathbf{v}_{\mathrm{H}}\frac{\partial p}{\partial \eta}\right)d\eta$$

Surface pressure

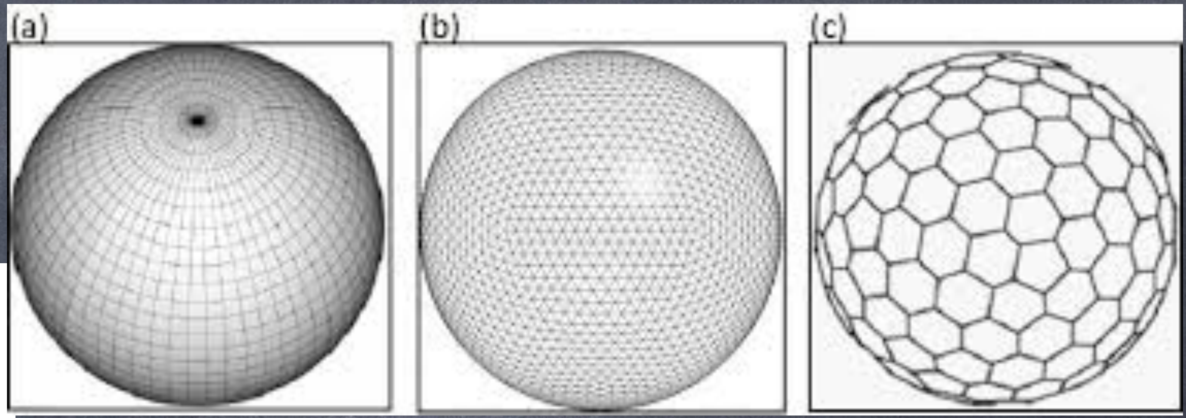
Got Grids?



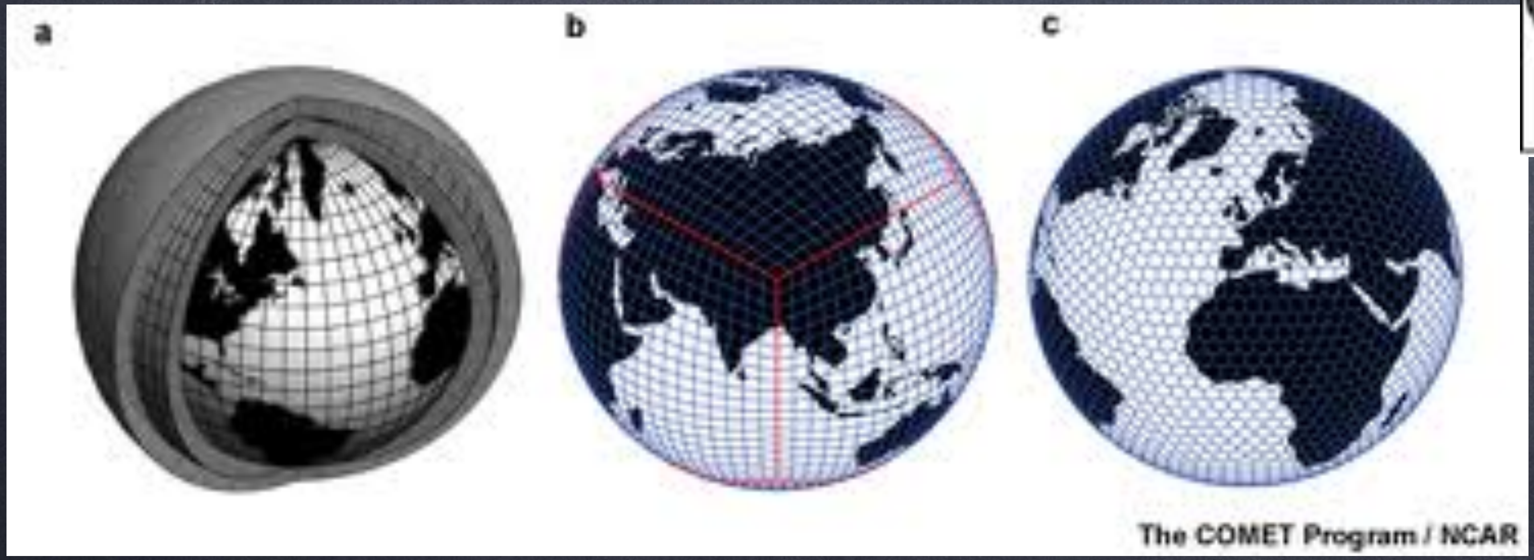
p
x
y
Δp
Δy
Δx
(x, y, p)
"grid point"
Δx × Δy × Δp
"grid volume"



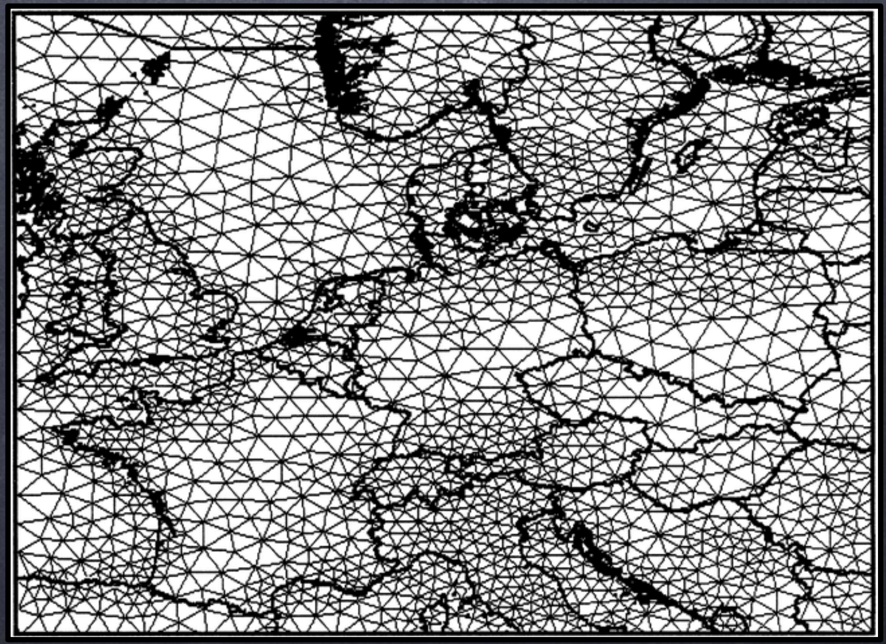
Yin
Yang



(a)
(b)
(c)



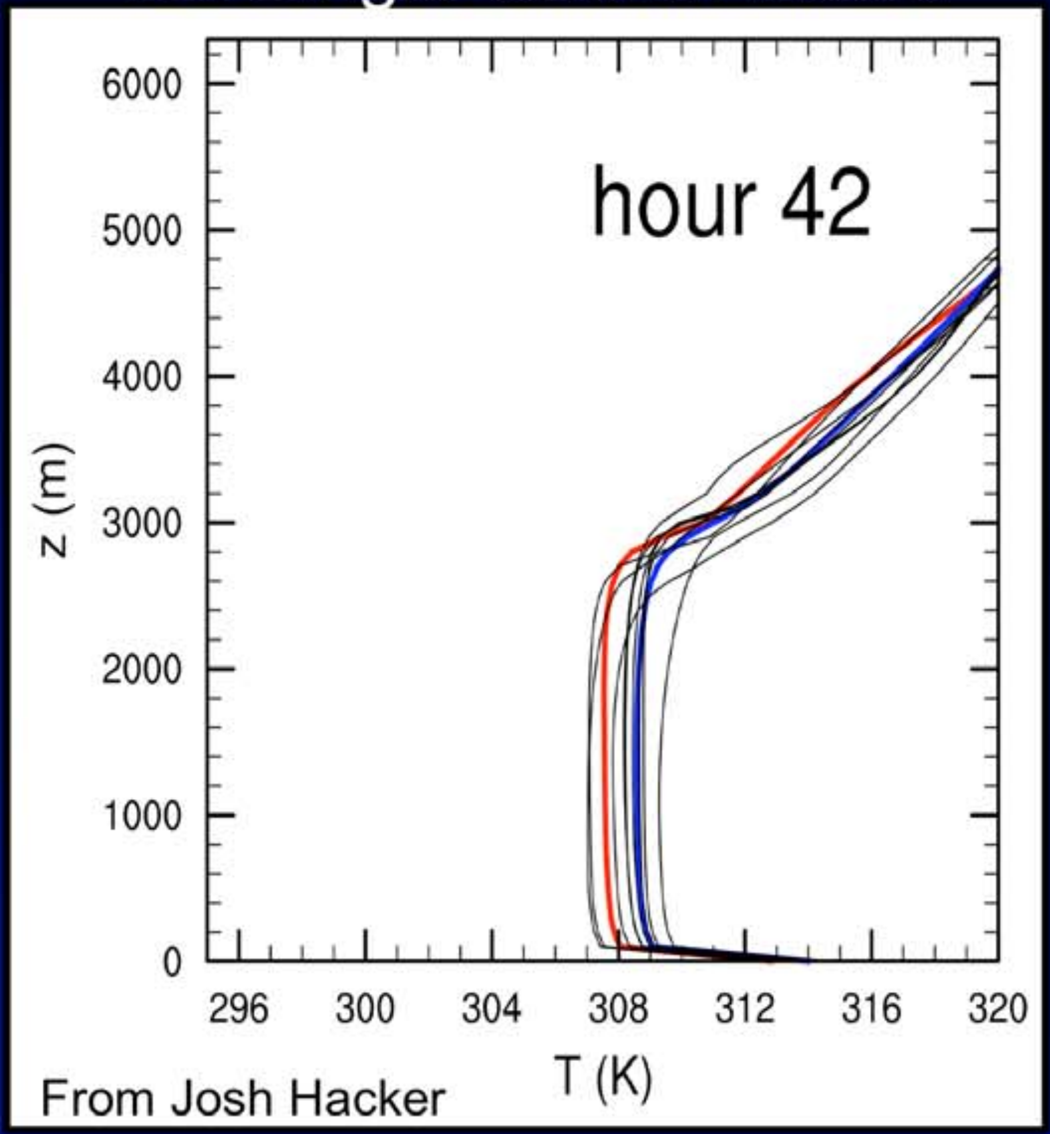
a
b
c
The COMET Program / NCAR



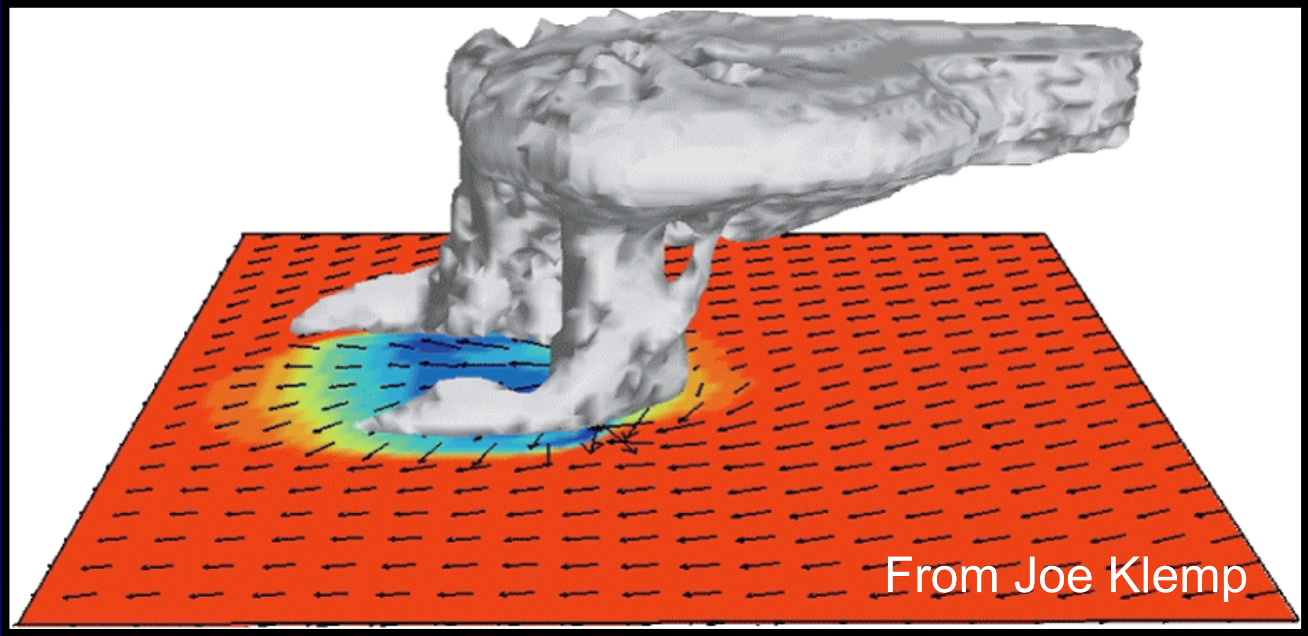
Domains

- Number of dimensions

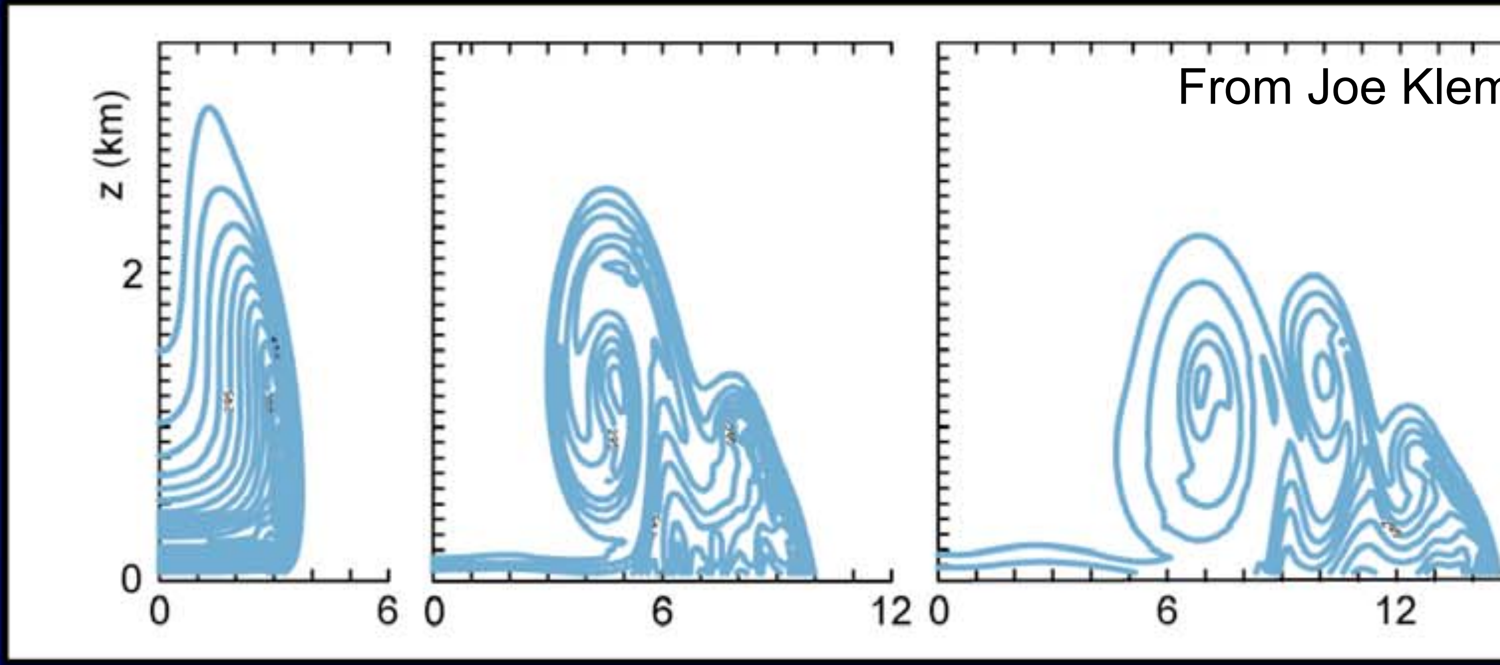
1D: Single-column model



3D: Simulation of thunderstorm



2D: Simulation of density current



Domains

- Number of dimensions
- Degrees of freedom and complexity of grid structure
- Shape (global, hemispheric, limited area)
- Vertical coordinate
- resolution

Typical Horizontal Grids

- Global Model
  - points
  - volumes (squares, hexagons, triangles, pentagons)
  - spherical harmonics
- Limited area
  - grid points
  - grid volumes (squares, hexagons, triangles, pentagons)
  - element based methods (spectral elements and DG)
- Each has various approximation methods (numerical methods)
  - grid points/grid volumes: local methods, easy to use distributed memory computers
  - spherical harmonics/element based methods: "global" methods, more communication needed (DG is an exception)

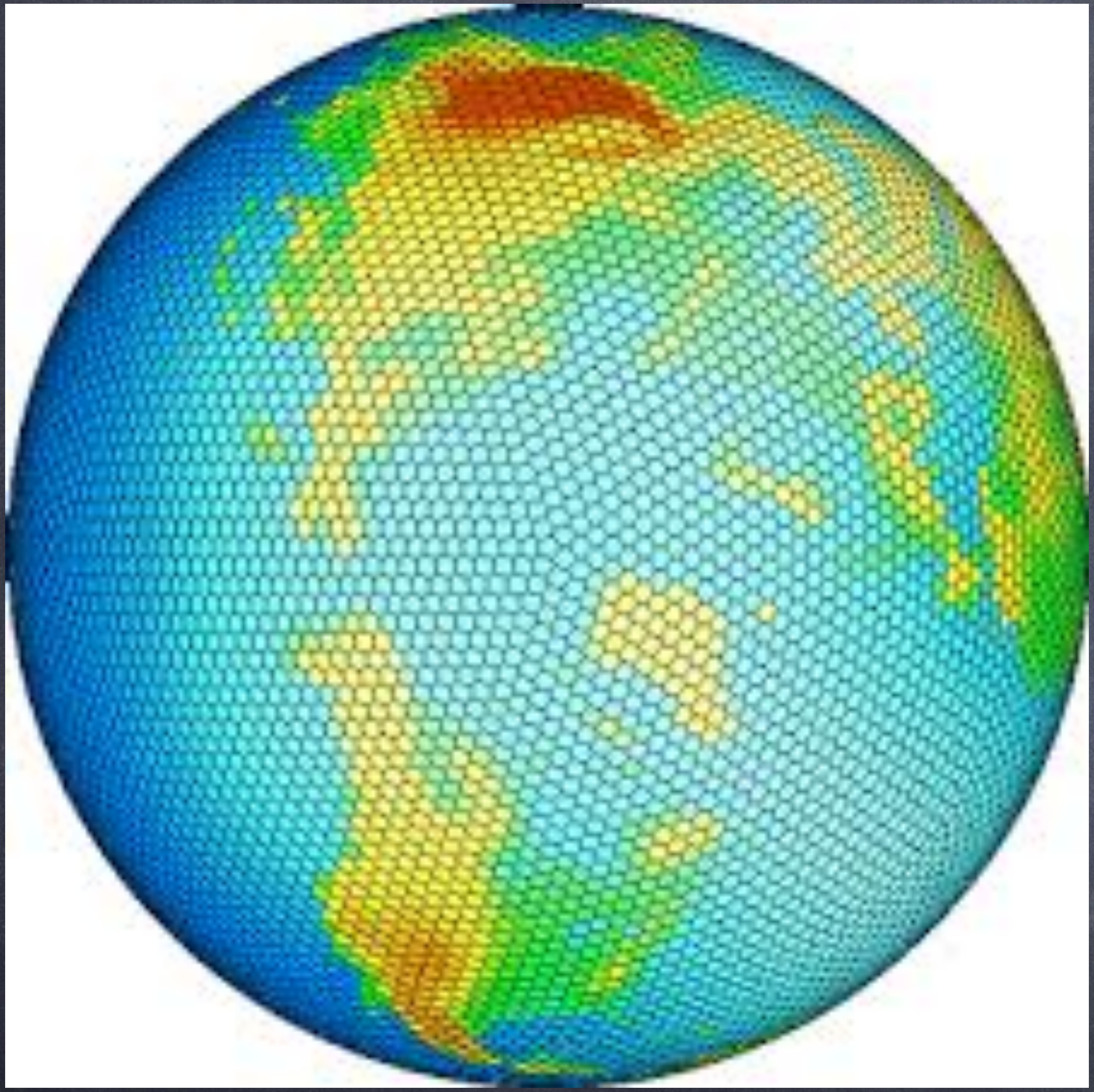
Structured Finite Diff grids

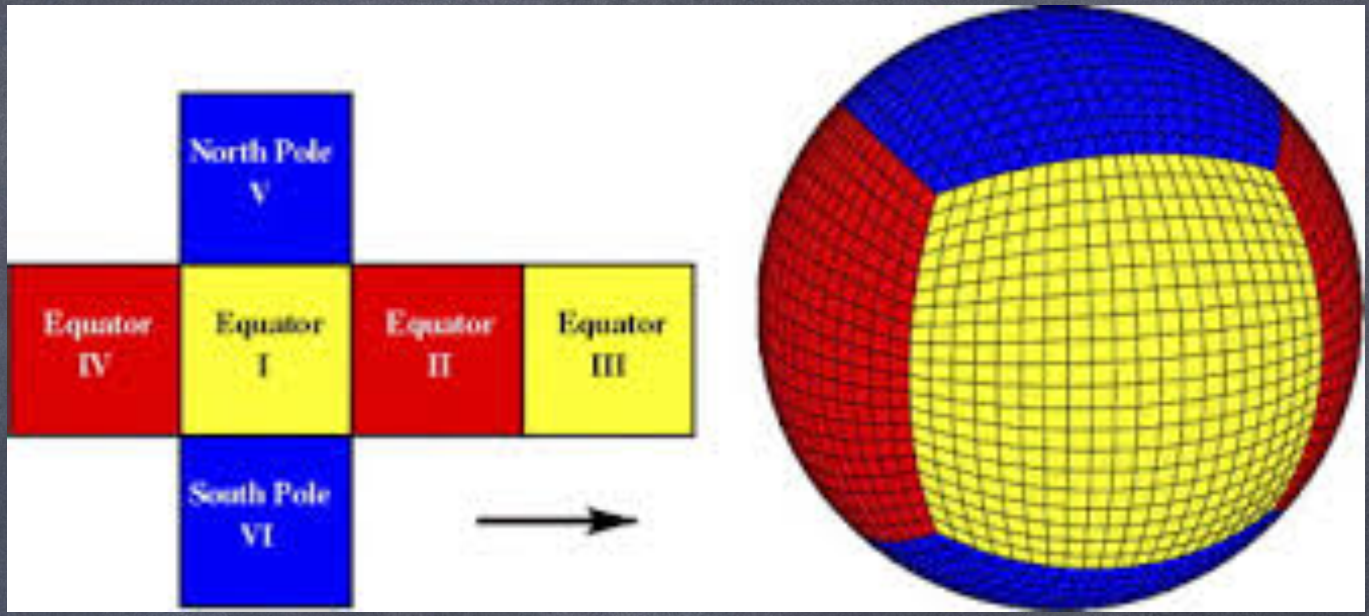
Lat/Lon Grid

- grid lines follow lat/lon
- what happens to dx at pole?
- Flow is same at pole as mid-lat..
- NWP trick: filtering of fields along longitudes at high-lats makes dx effectively larger... CFL stability is then maintained

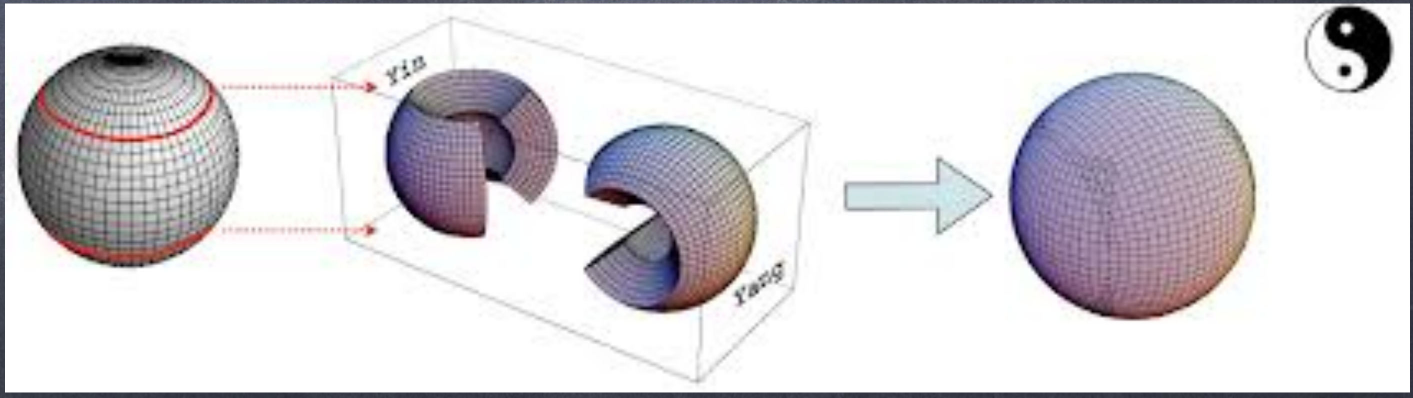
These models going out of production: too inefficient

Structured Finite Vol grids





FV-Core GFDL

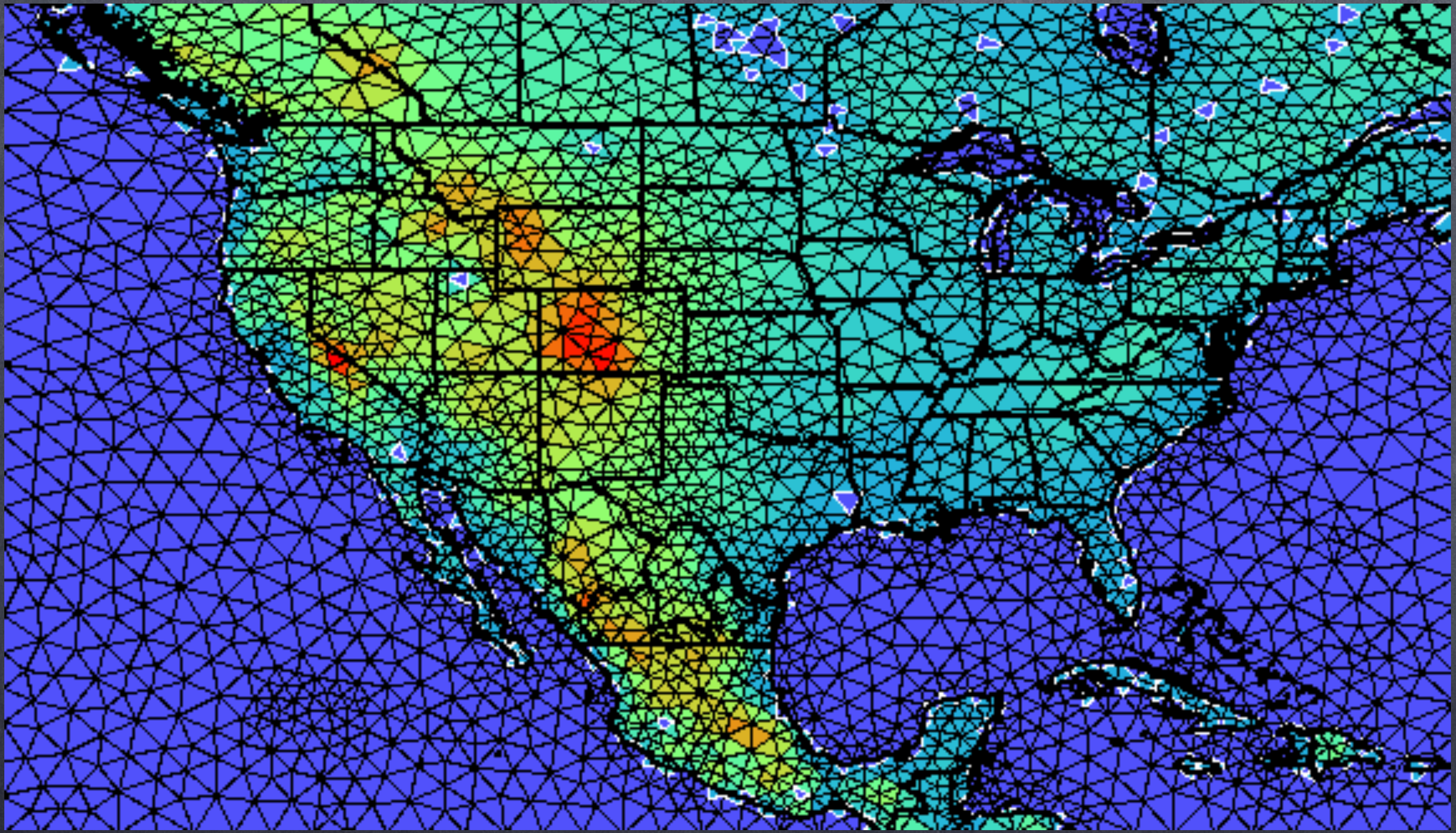


NICAM
(Japan)

Simple data structures
non-adaptive
run FAST

Yin-Yang
Japan

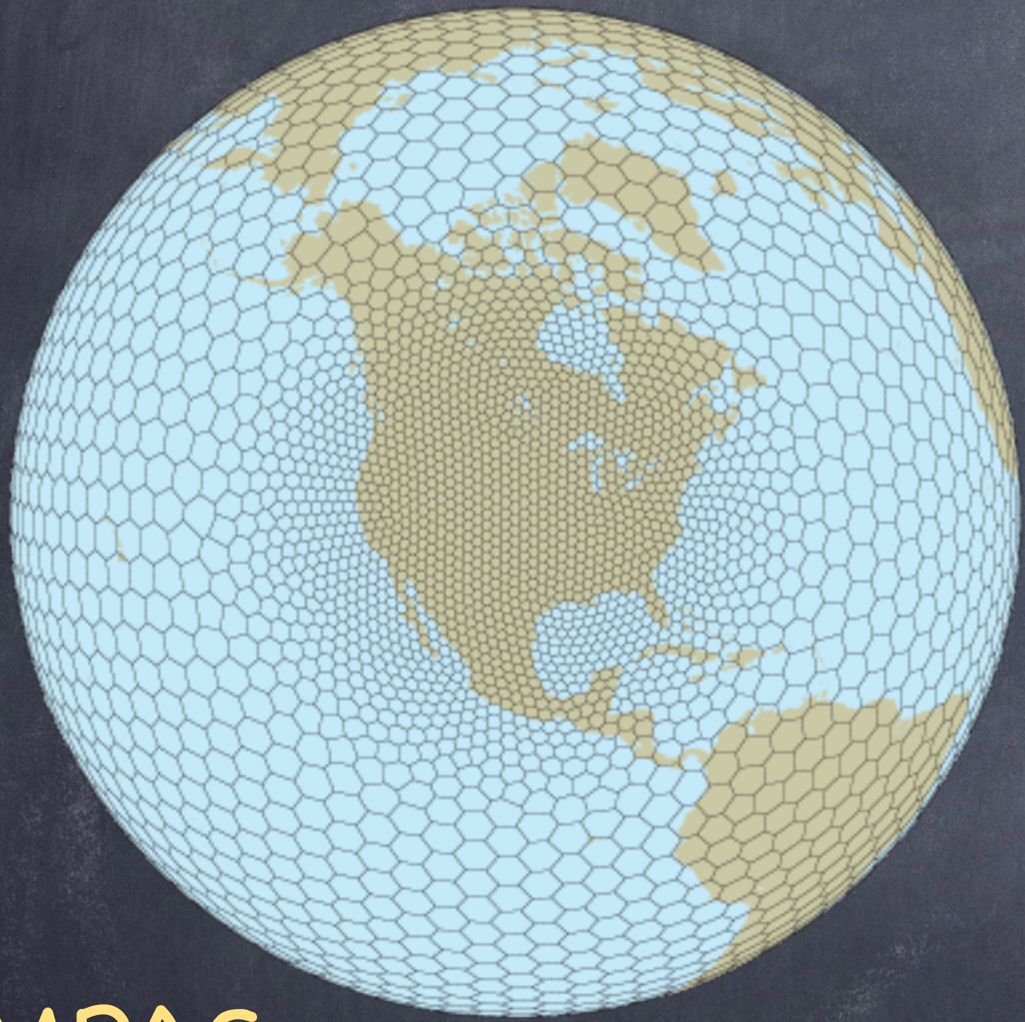
UnStructured Finite Element grids



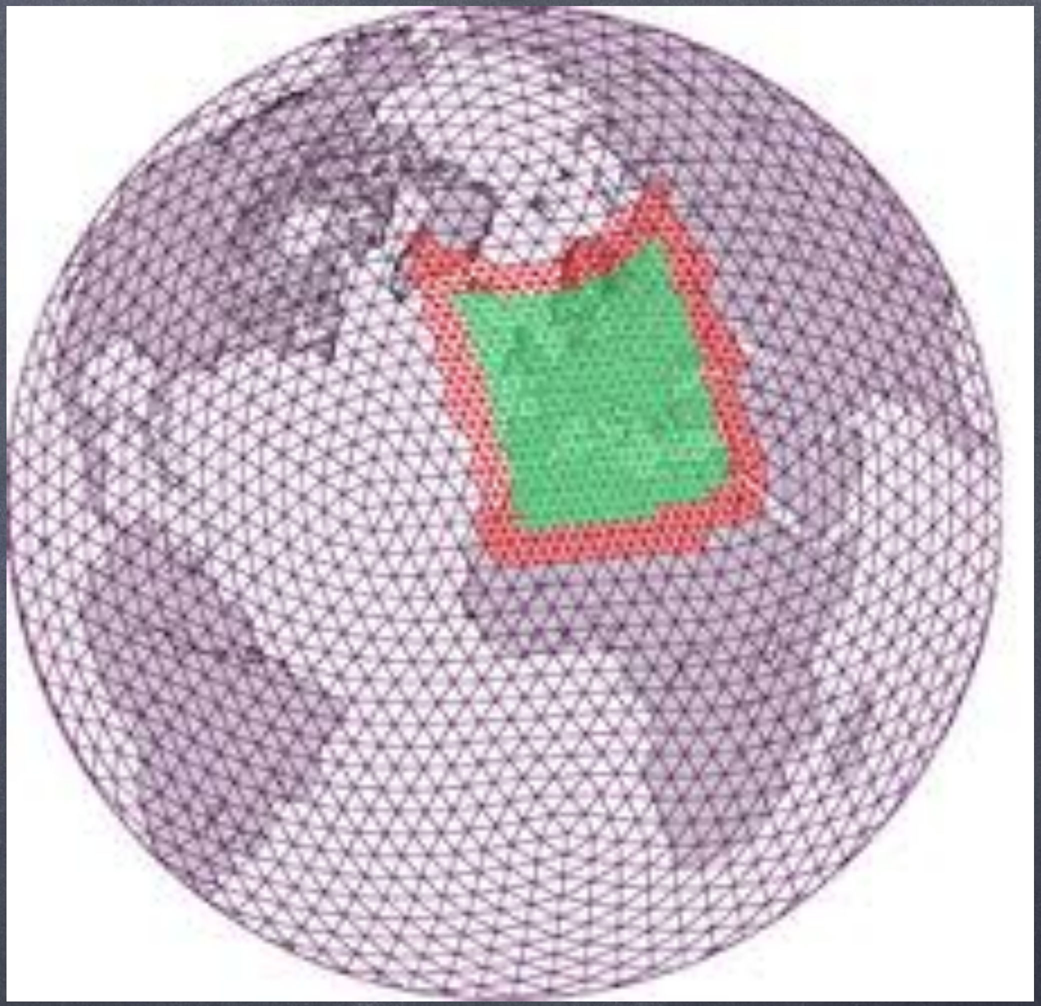
OMEGA Model

complicated data structure
Fully adaptive
more difficult to run fast

UnStructured Finite Volume grids



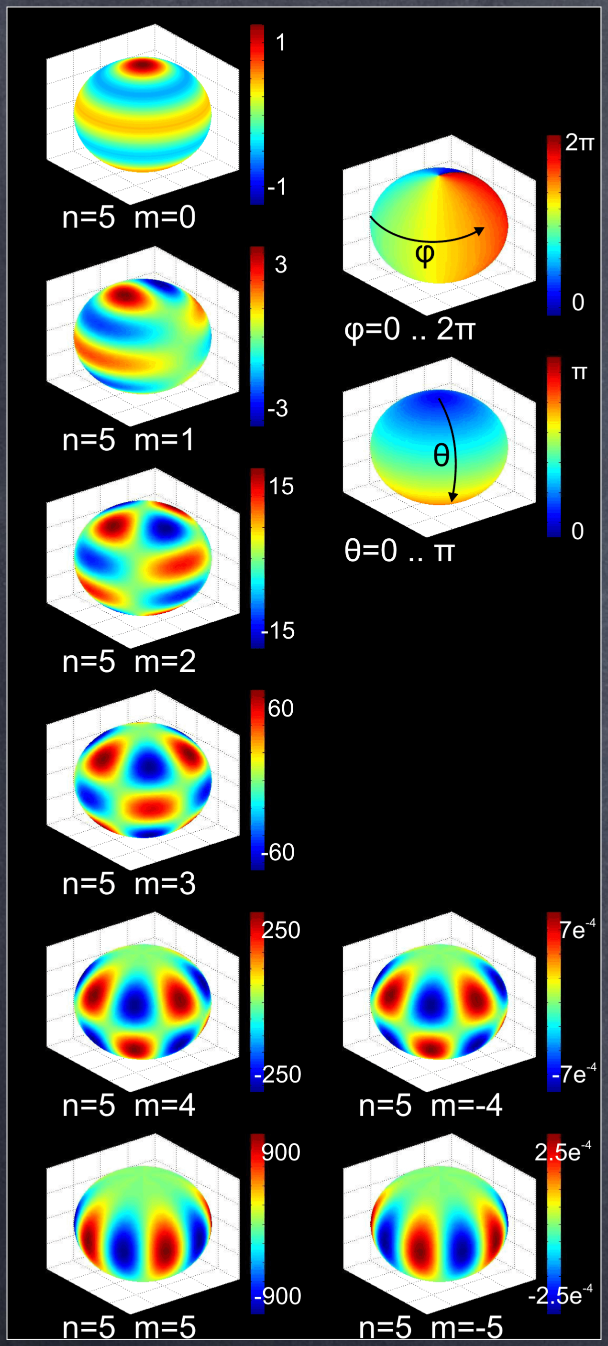
MPAS
(NCAR)

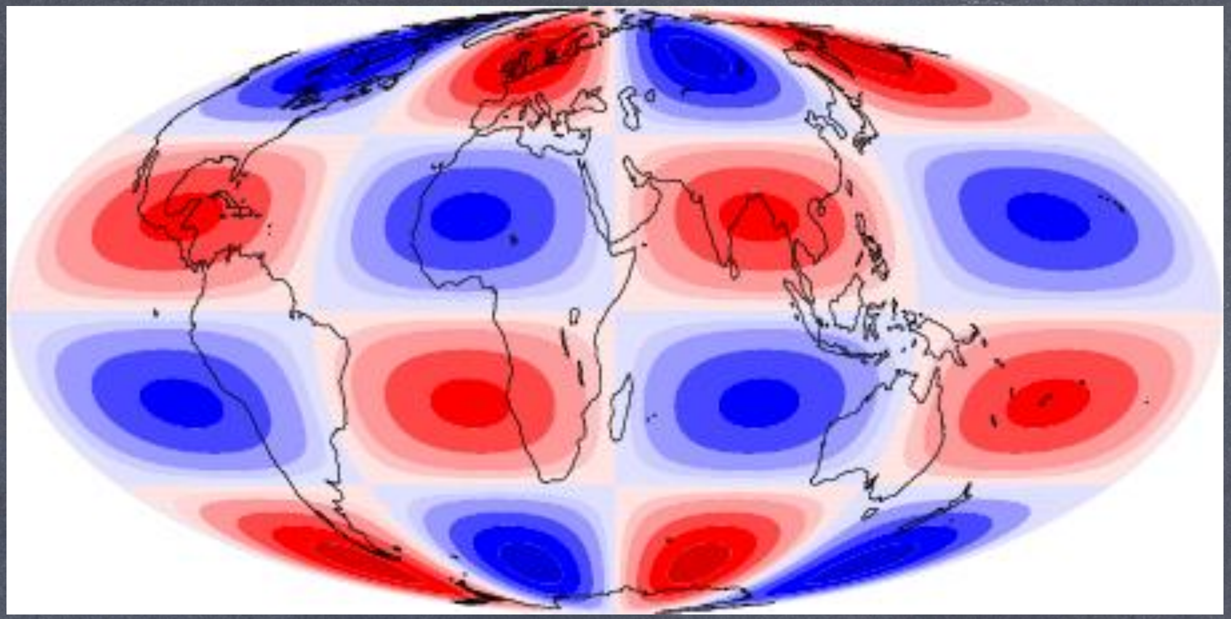


ICON
(DWD)

moderately cmplx data structure
non-time dependent adaptive
runs fast!

Spectral Grids





East-West: Fourier Series

North-South: Legendre Polynomials

$$T(\lambda,\mu,t) = \sum_{m=-M}^{M} \sum_{n=|m|}^{N(m)} T(t) P_n^m(\mu) e^{im\lambda}$$

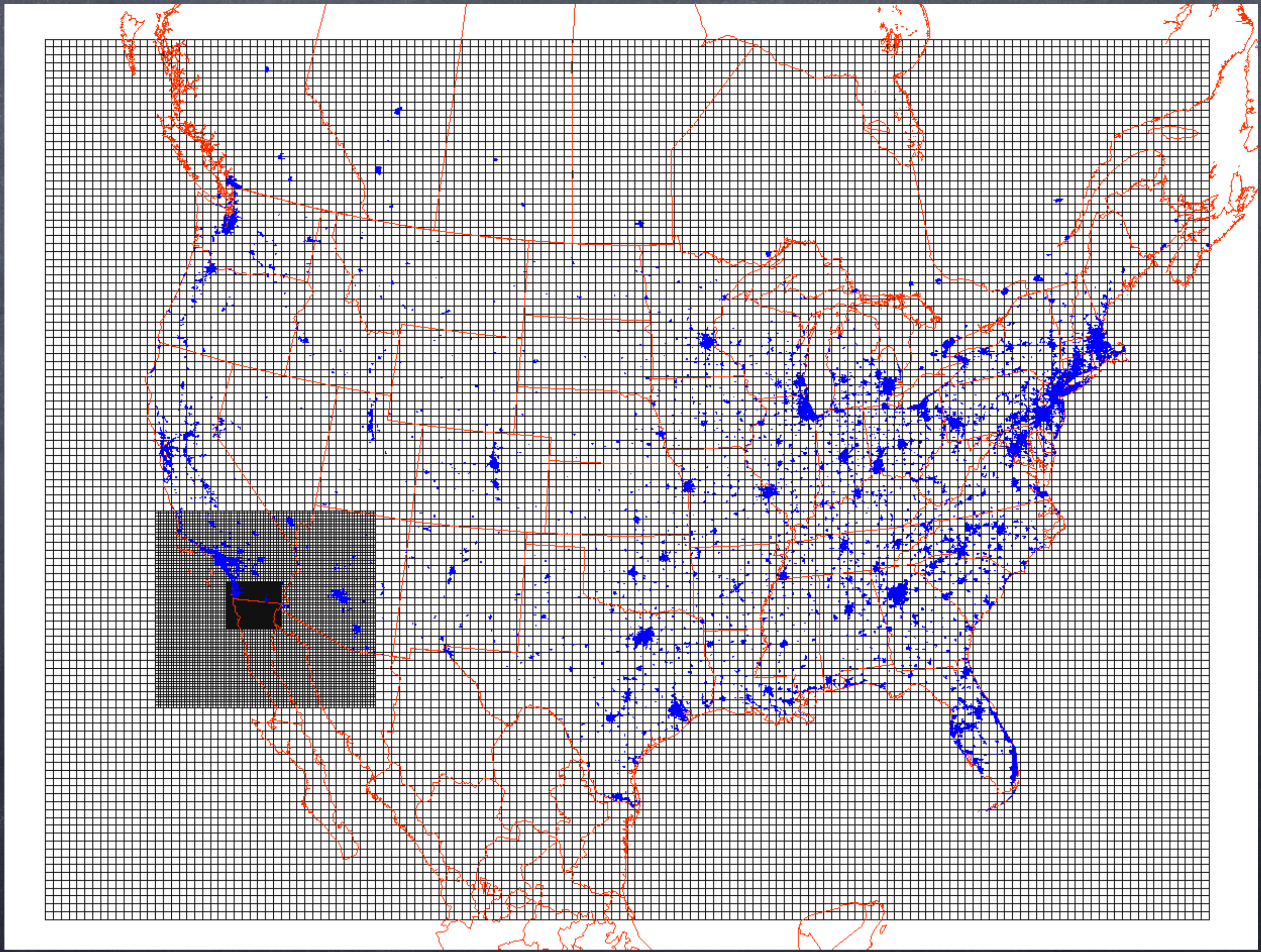
$\lambda = longitude$

$\mu = \sin(\phi)$

Predicting amplitudes
T(t)

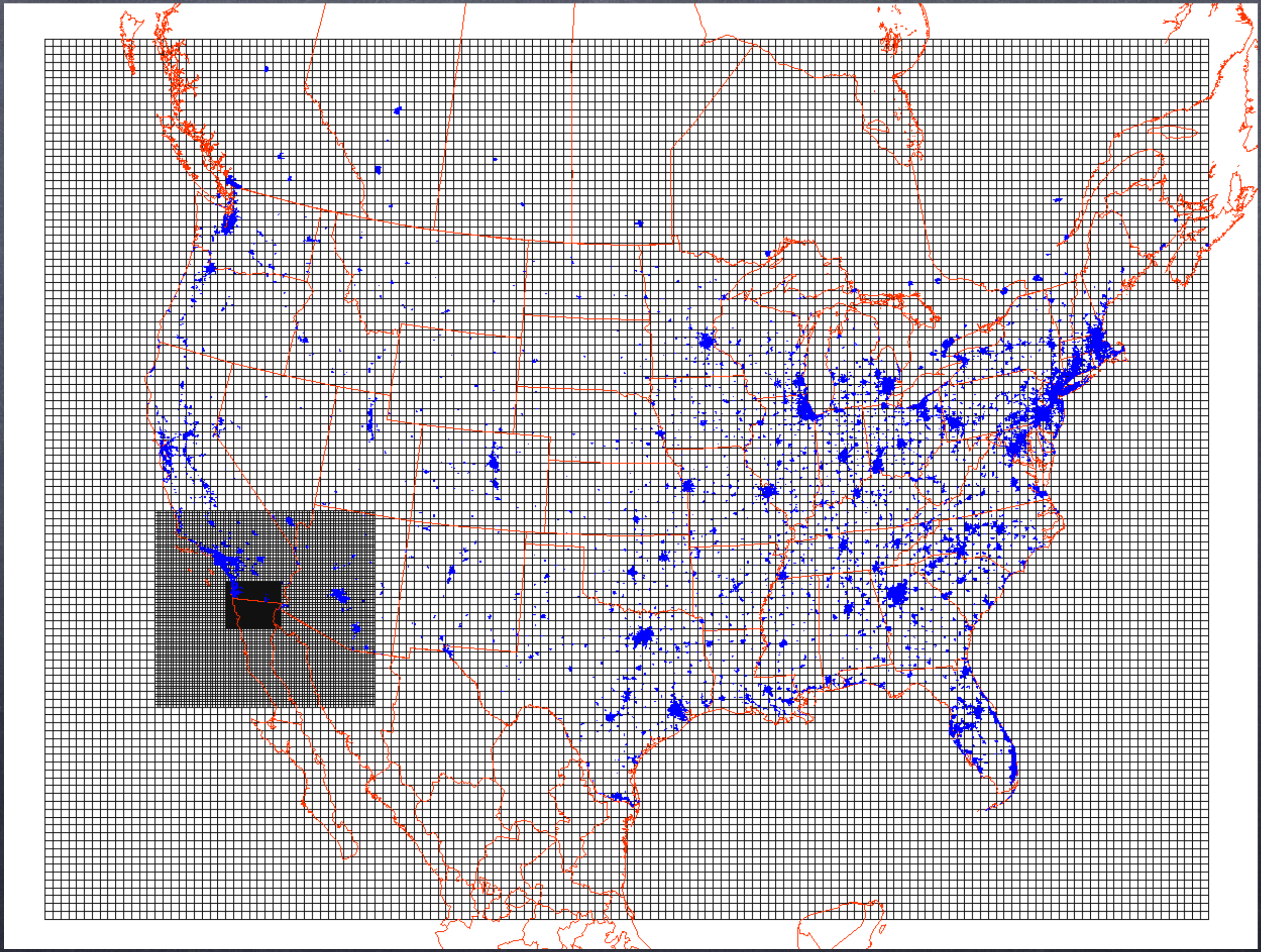
Limited Area Models

- 3 Grids
- One large domain
- 2 smaller domains



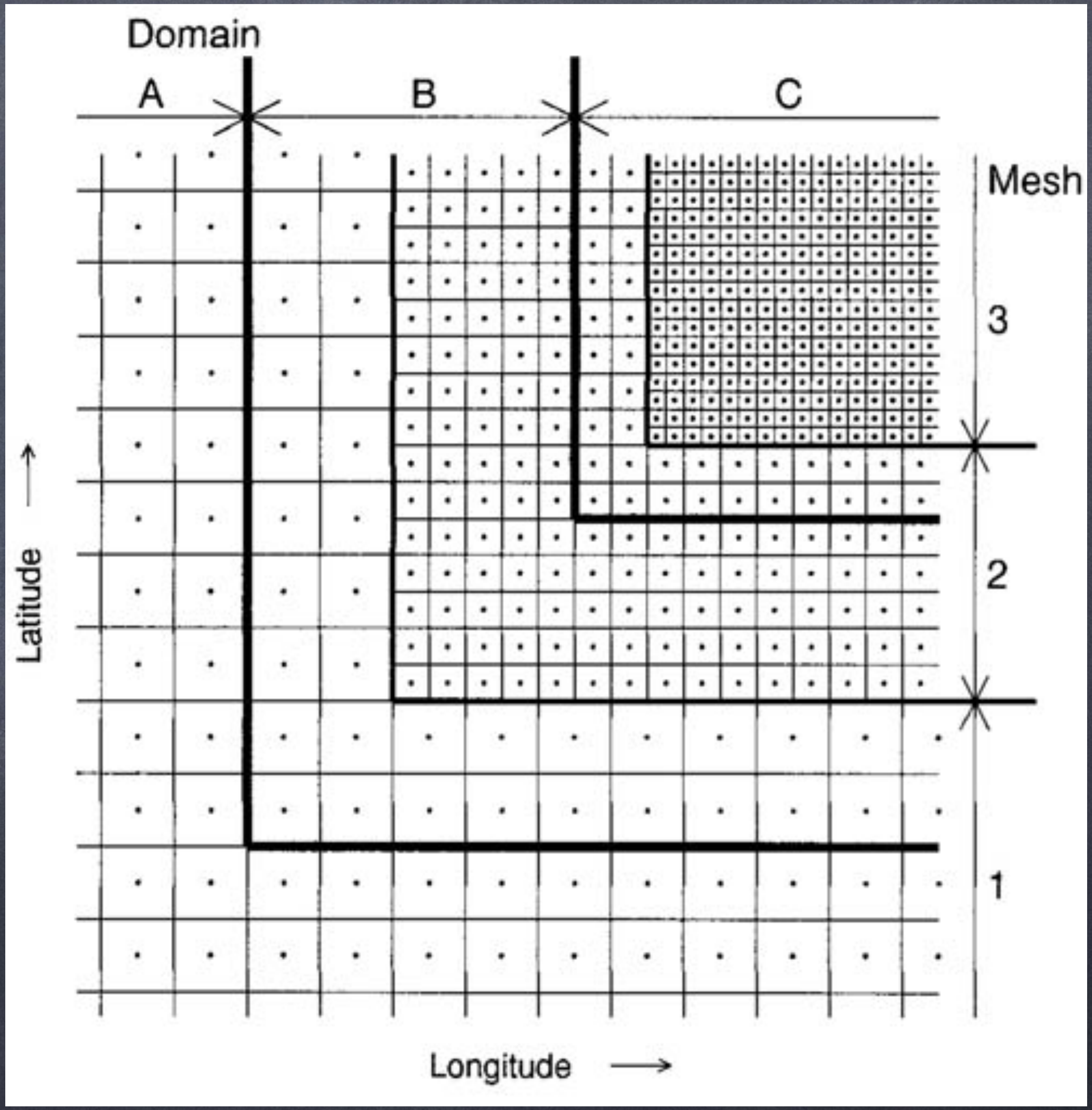
Limited Area Models

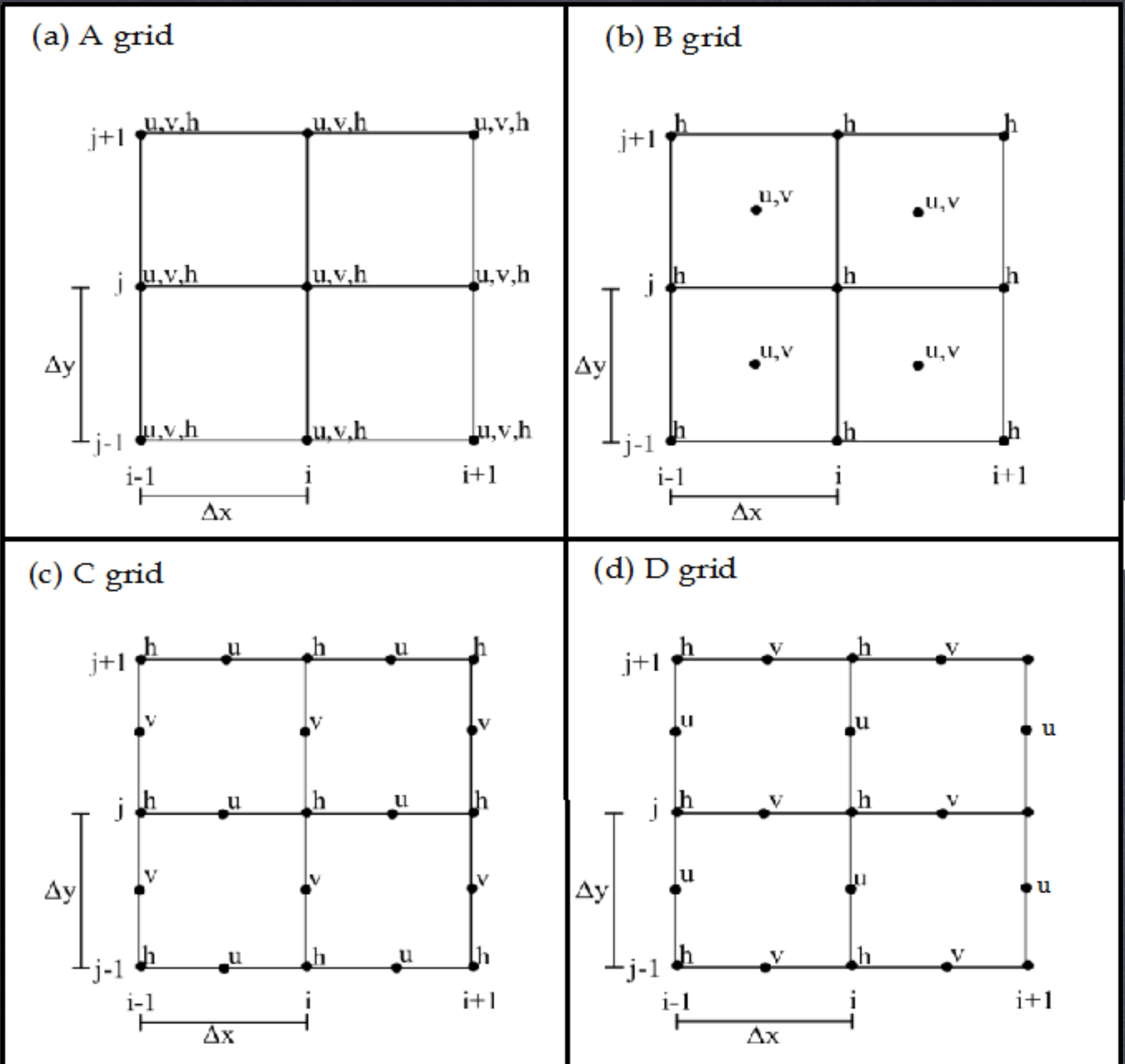
- Need initial conditions AND
- Boundary conditions!



Limited Area Models

- Nested grid boundary conditions are tricky between grids
- Abrupt changes in resolution create jumps in solution - generate noise!





Staggered Grids

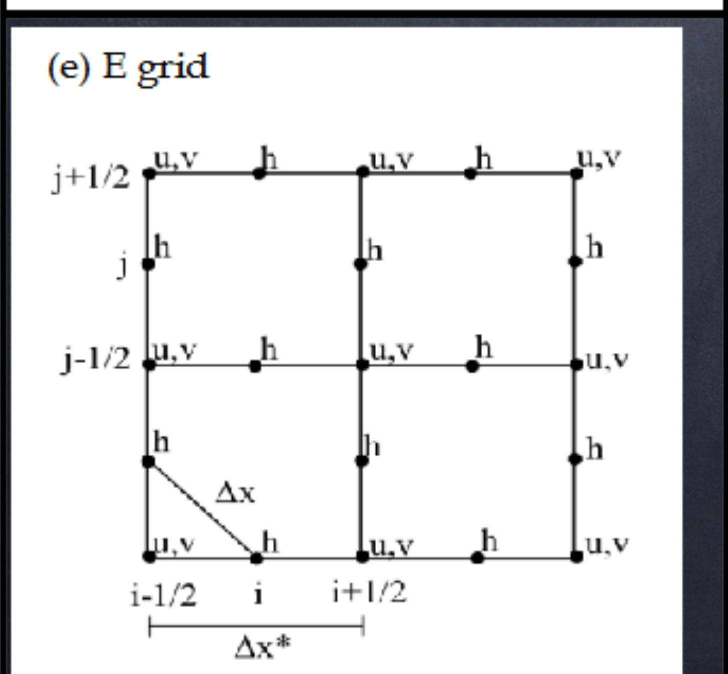
A = unstaggered

B = U/V in corner

C = U/V on edges

D = V/U on edges

E = 45 deg B-grid



- Most models stagger the velocity variables
- Used to increase the computational accuracy of derivatives associated with the divergence and PGF terms
- Staggering is also used in the vertical
- While inconvenient, staggered grids INCREASE the accuracy of the numerical solution (so much so you really cannot get away from it….) for almost no cost

Vertical Coordinate Systems

- Height
- Pressure
- Sigma
- ETA
- Isentropic
- Hybrids

Domains

- Vertical coordinate

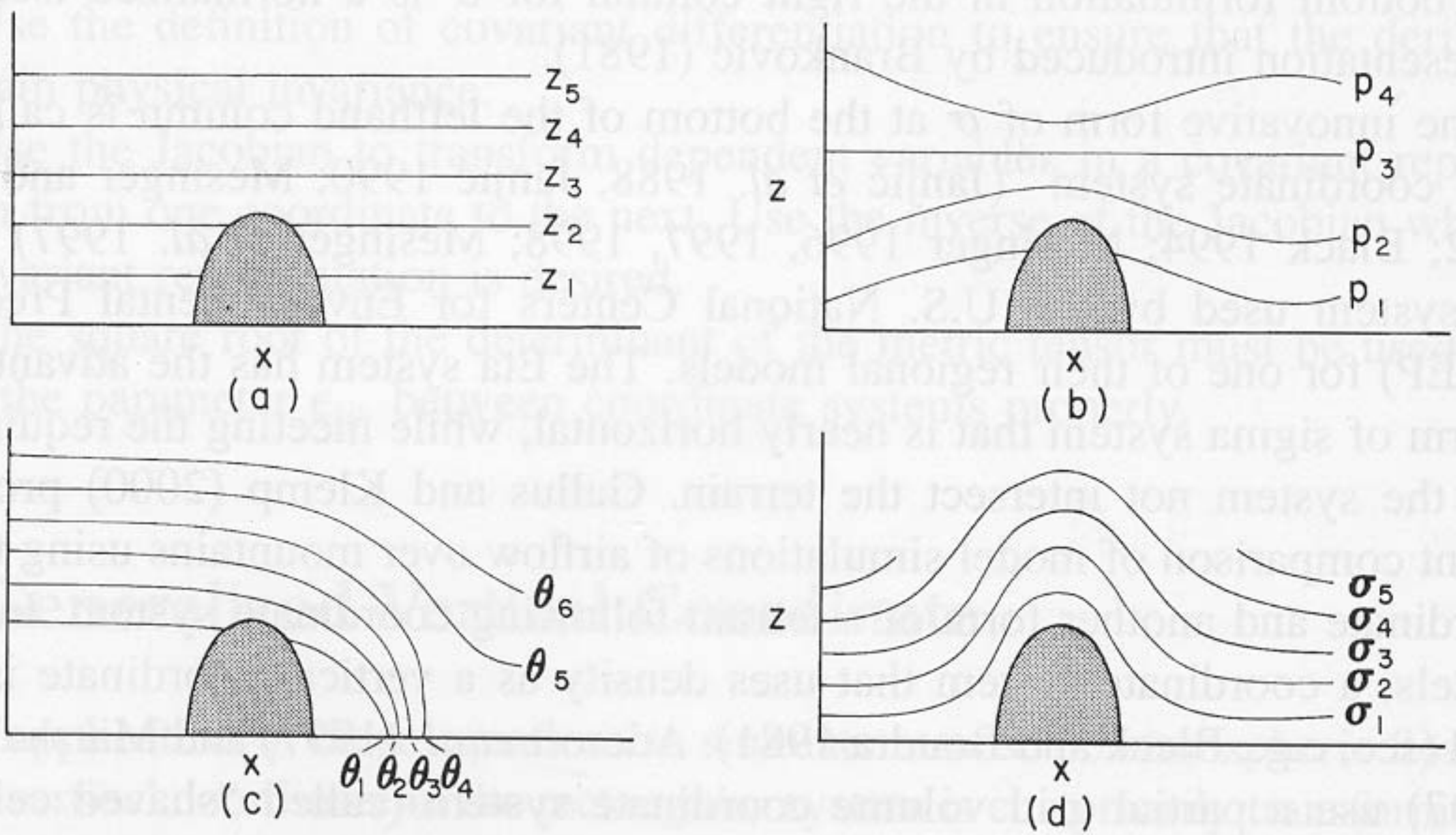
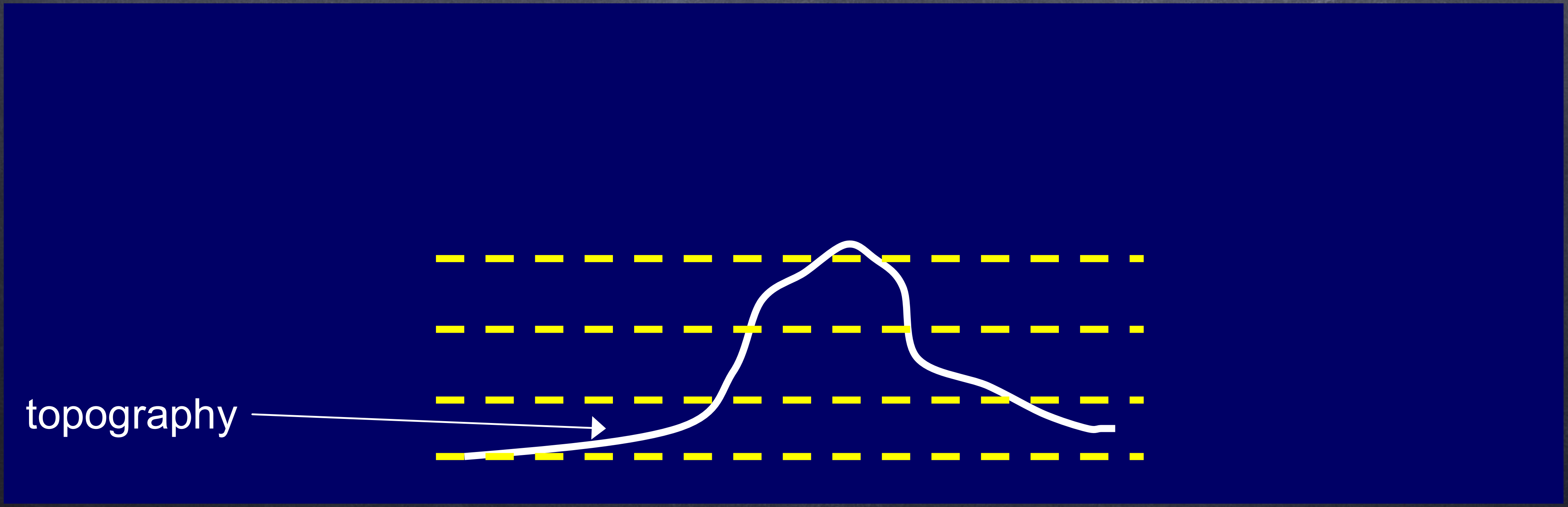


Fig. 6-2. Schematic illustrations of (a) rectangular, (b) isobaric, (c) isentropic, and (d) sigma coordinate representations as viewed in a rectangular coordinate framework.

From Pielke (2002)

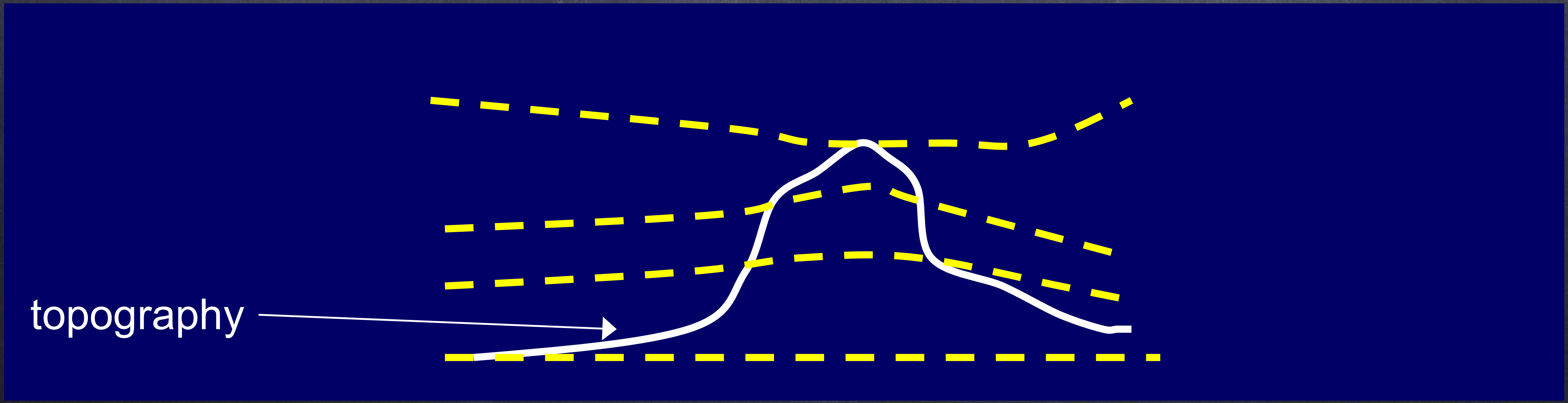
Height as a Vertical Coordinate

- Advantages
  - easy, intuitive
- Disadvantages
  - topography hard to deal with…



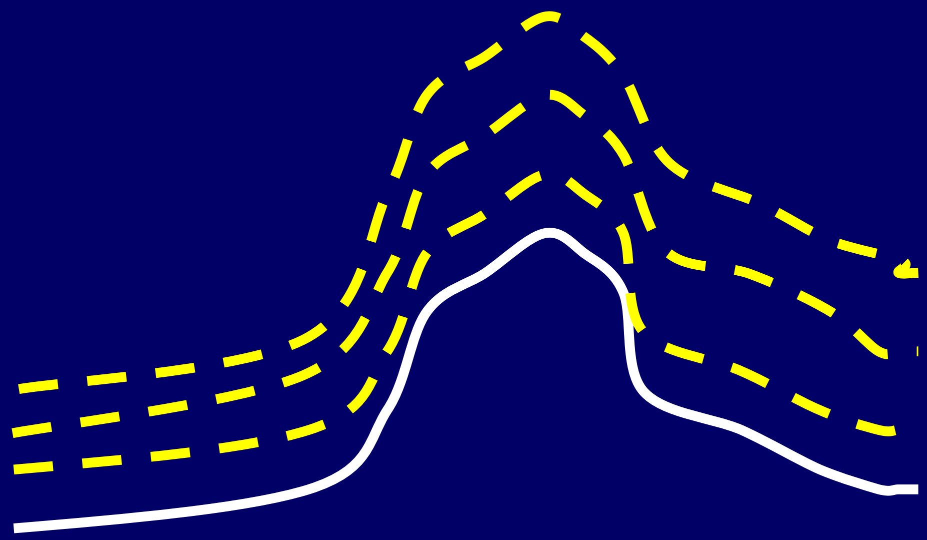
Pressure as a Vertical Coordinate

- Advantages
  - top of atmosphere is easy (p=0)
  - observations often in terms of pressure (rawinsonde, satellite)
- Disadvantages
  - pressure has same problems as height.



Sigma as a Vertical Coordinate

- Advantages: easy to represent top and bottom of atmosphere
- Disadvantages: equations need to be transformed, errors in horizontal PGF when terrain slope is steep



$$\sigma = \frac{p}{p_{sfc}}$$

•Terrain following vertical coordinate.
•Sigma = Pressure/Surface Pressure
•σ = 0 at the top of the atmosphere.
•σ = 1 at the Earth's surface.

A brief foray into Coordinate Transformations

$$A(x,y,z,t) = A(x,y,\zeta(x,y,z,t),t)$$

vert $\dfrac{\partial A}{\partial \zeta} = \dfrac{\partial z}{\partial \zeta}\dfrac{\partial A}{\partial z}$

trans $\dfrac{\partial A}{\partial z} = \dfrac{\partial \zeta}{\partial z}\dfrac{\partial A}{\partial \zeta}$

grad $\nabla_z A = \nabla_\zeta A - \dfrac{\partial A}{\partial \zeta}\dfrac{\partial \zeta}{\partial z}\nabla_\zeta z$

div $\nabla_z \bullet \vec{A} = \nabla_\zeta \bullet \vec{A} - \dfrac{\partial \vec{A}}{\partial \zeta}\dfrac{\partial \zeta}{\partial z}\nabla_\zeta z$

let $s \rightarrow x,y,t$

$$\left.\frac{\partial A}{\partial s}\right|_\zeta = \left.\frac{\partial A}{\partial s}\right|_z + \frac{\partial A}{\partial \zeta}\frac{\partial \zeta}{\partial z}\left.\frac{\partial A}{\partial \zeta}\right|_\zeta$$

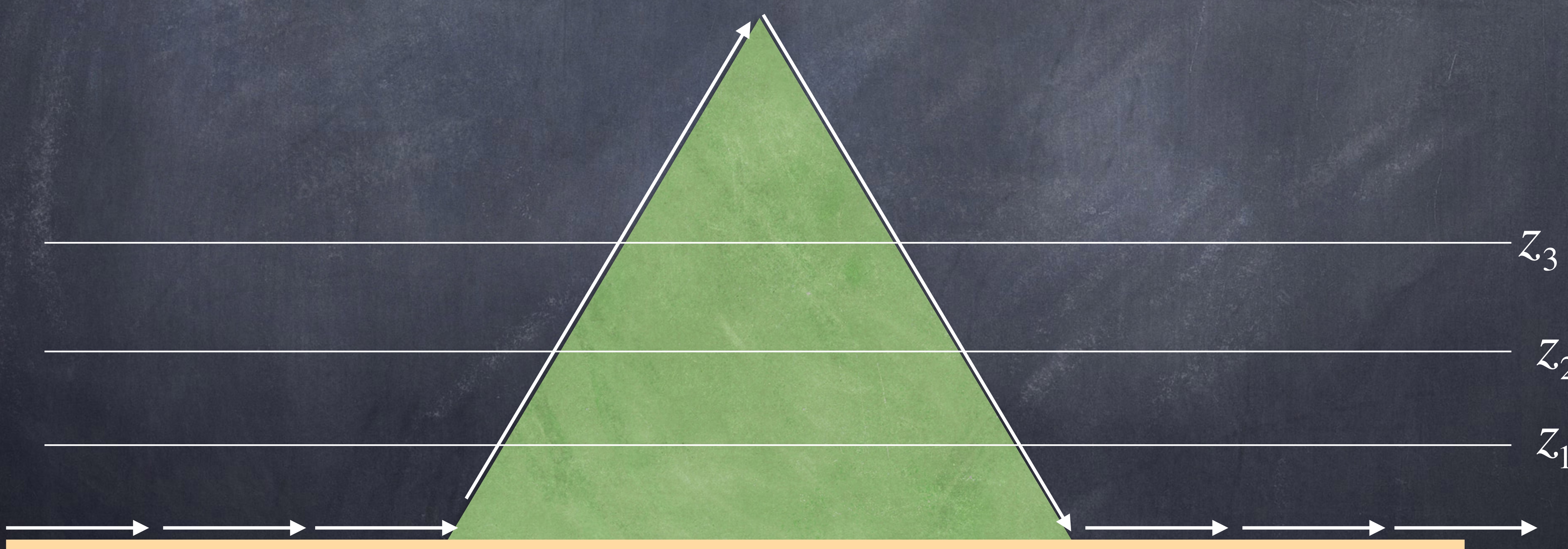
vert velo

$$w = D_t z = \left.\frac{\partial z}{\partial t}\right|_\zeta + \left.\frac{\partial z}{\partial x}\right|_\zeta D_t x + \left.\frac{\partial z}{\partial y}\right|_\zeta D_t y + \left.\frac{\partial z}{\partial \zeta}\right|_\zeta D_t \zeta$$

$$w = \left.\frac{\partial z}{\partial t}\right|_\zeta + \vec{v}_h \bullet \nabla_\zeta z + \omega \left.\frac{\partial z}{\partial \zeta}\right|_\zeta$$

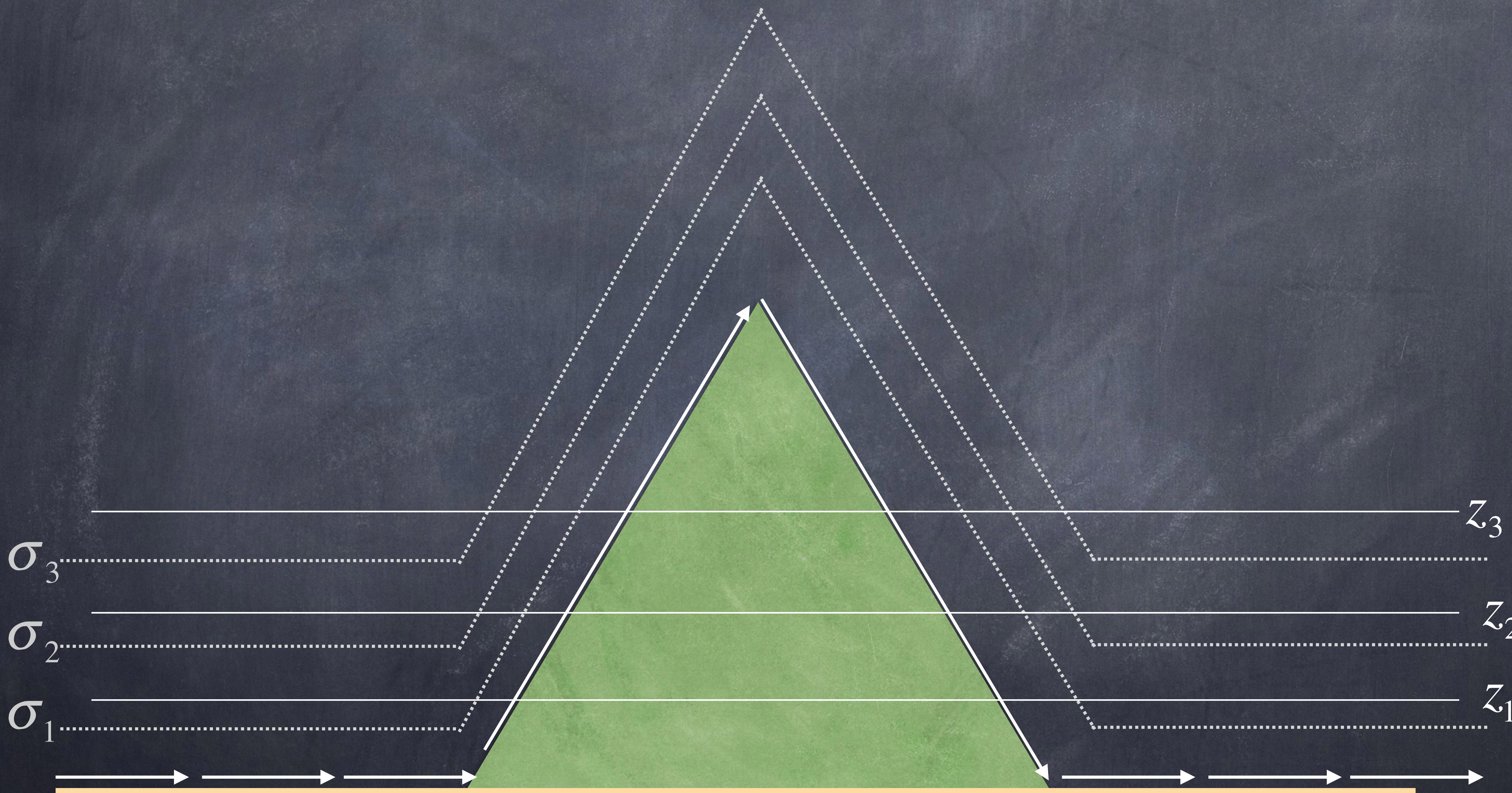
$$w = \cancel{\left.\frac{\partial z}{\partial t}\right|_\zeta} + \vec{v}_h \bullet \nabla_\zeta z + \omega \left.\frac{\partial z}{\partial \zeta}\right|_\zeta = \vec{v}_h \bullet \nabla_\zeta z + \omega \left.\frac{\partial z}{\partial \zeta}\right|_\zeta$$

Coord Trans Example



Flow along ground (u > 0)

Coord Trans Example

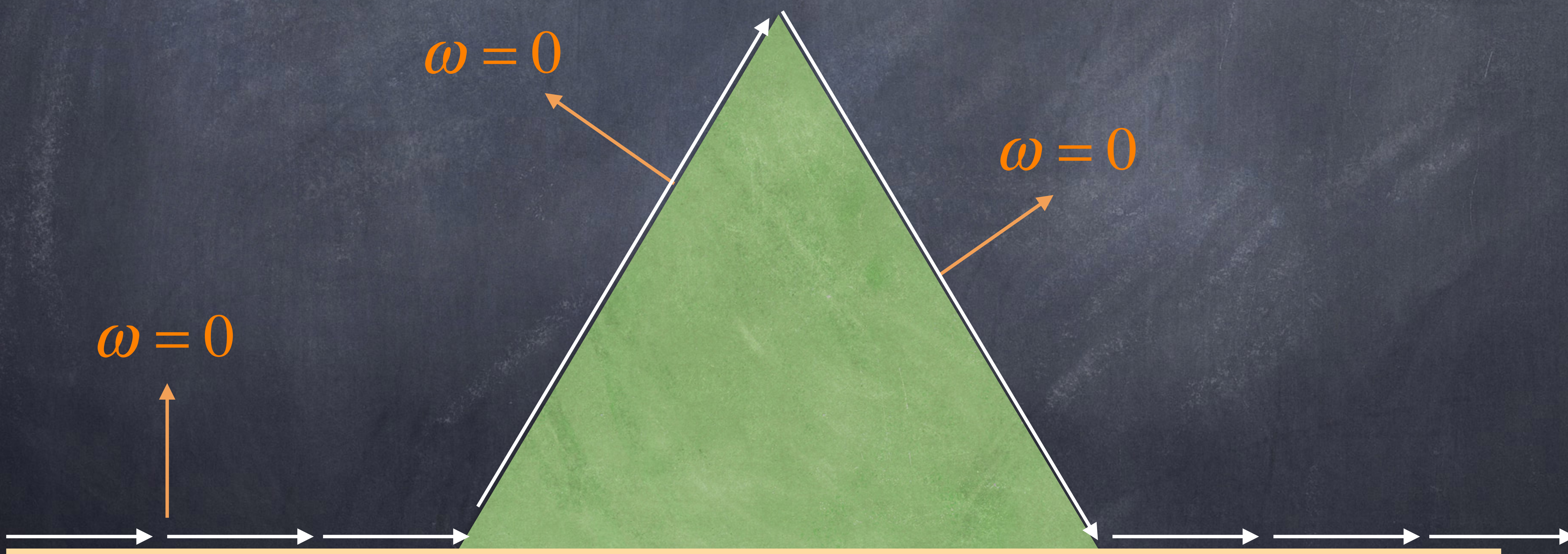


Flow along ground (u > 0)

Example

At ground omega = 0

$$w = \vec{v}_h \bullet \nabla_\zeta z + \omega \left.\frac{\partial z}{\partial \zeta}\right|_\zeta$$



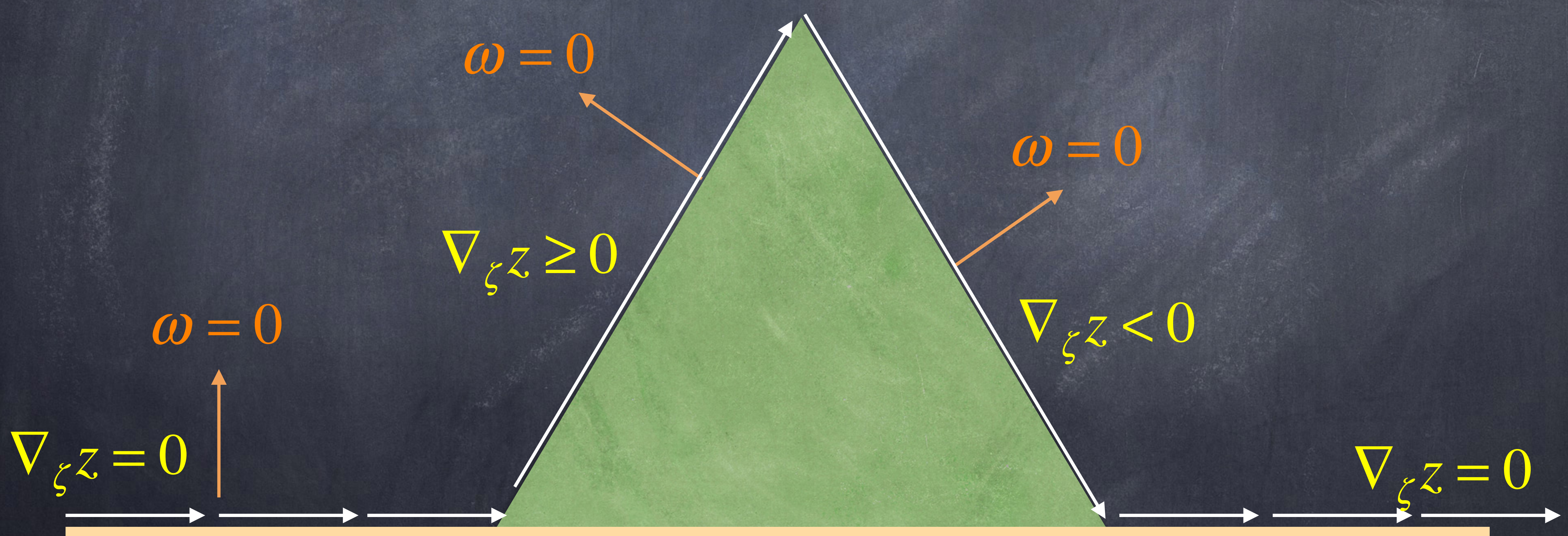
Flow along ground (u > 0)

Example

$$w = \vec{v}_h \cdot \nabla_\zeta z + \omega \left.\frac{\partial z}{\partial \zeta}\right|_\zeta$$

At ground omega = 0

grad(Z) ne 0!



Flow along ground (u > 0)

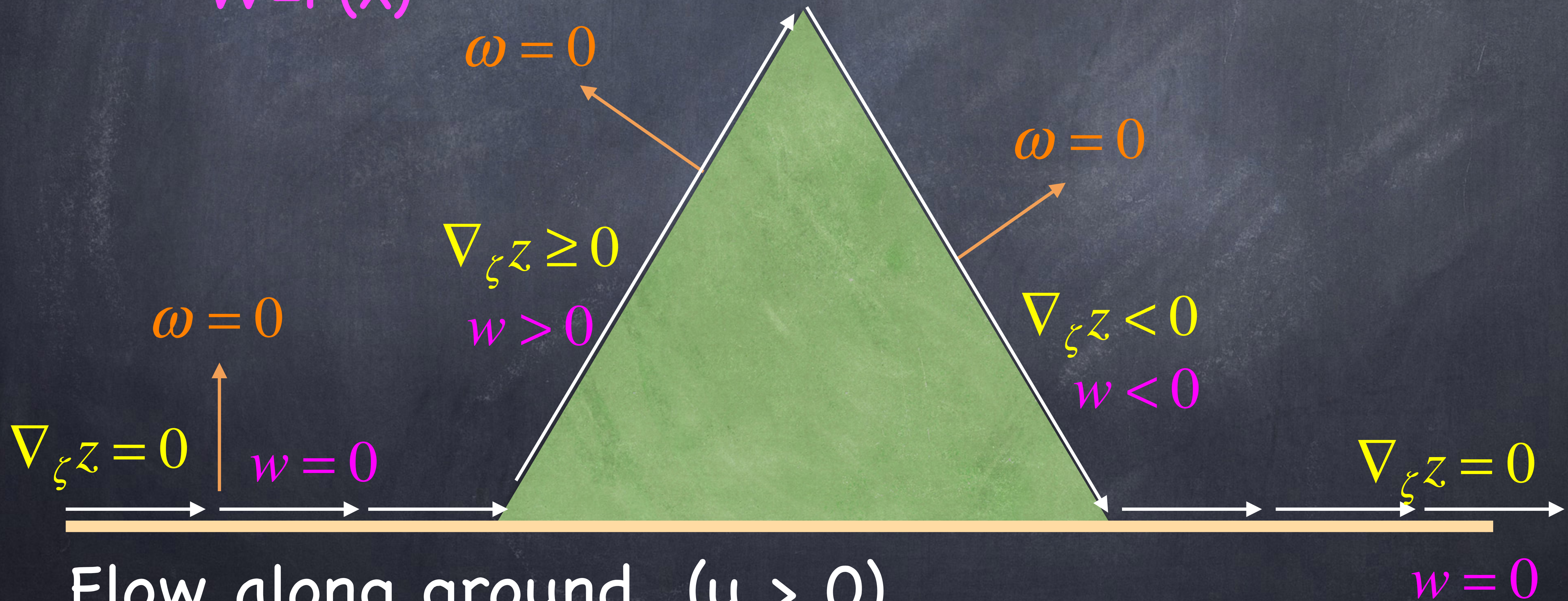
Example

$$w = \vec{v}_h \bullet \nabla_\zeta z + \omega \frac{\partial z}{\partial \zeta}\bigg|_\zeta$$

At ground omega = 0

grad(Z) ne 0!

W=F(x)



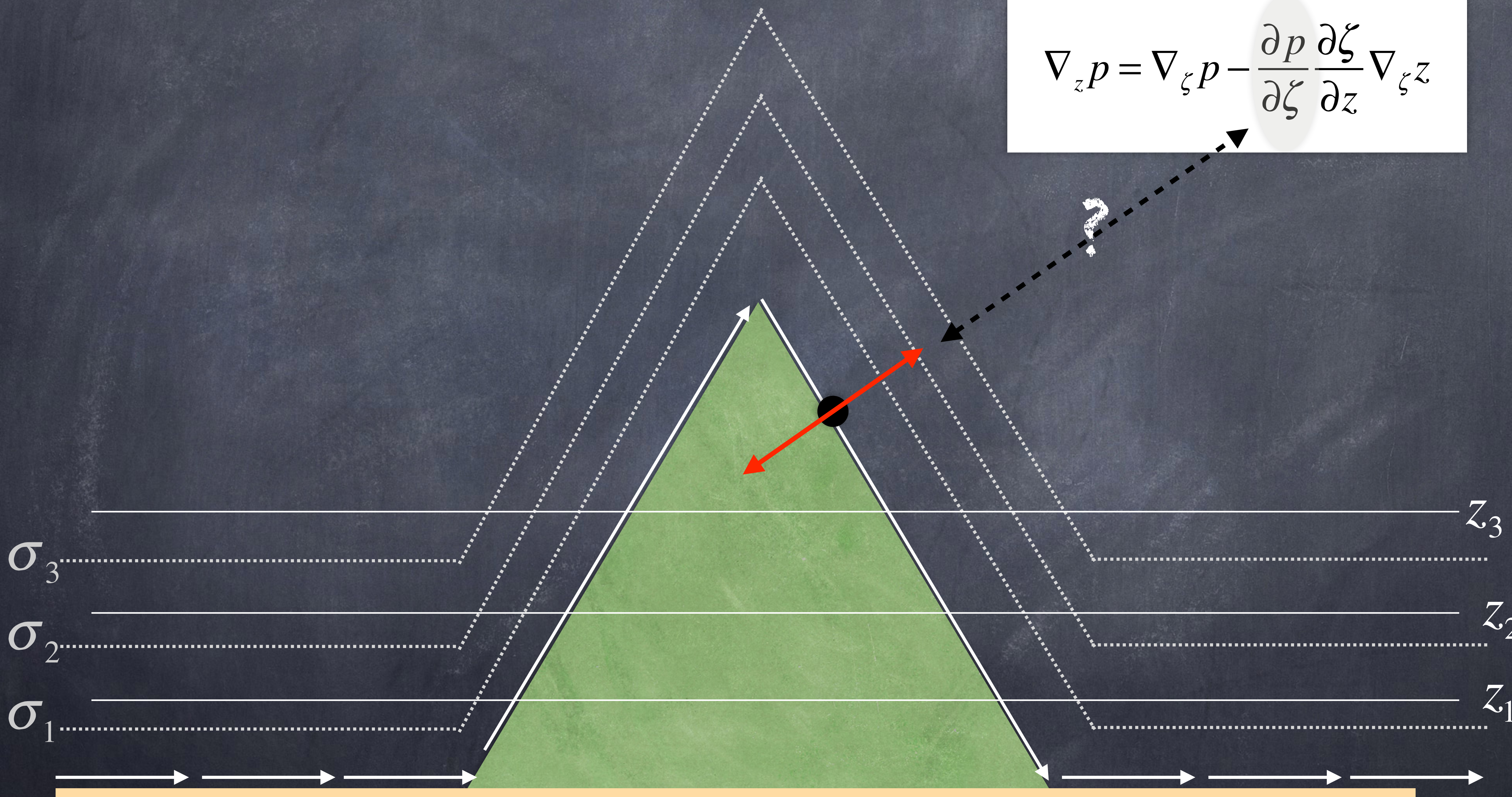
Flow along ground (u > 0)

Problems with Coord Transformations!

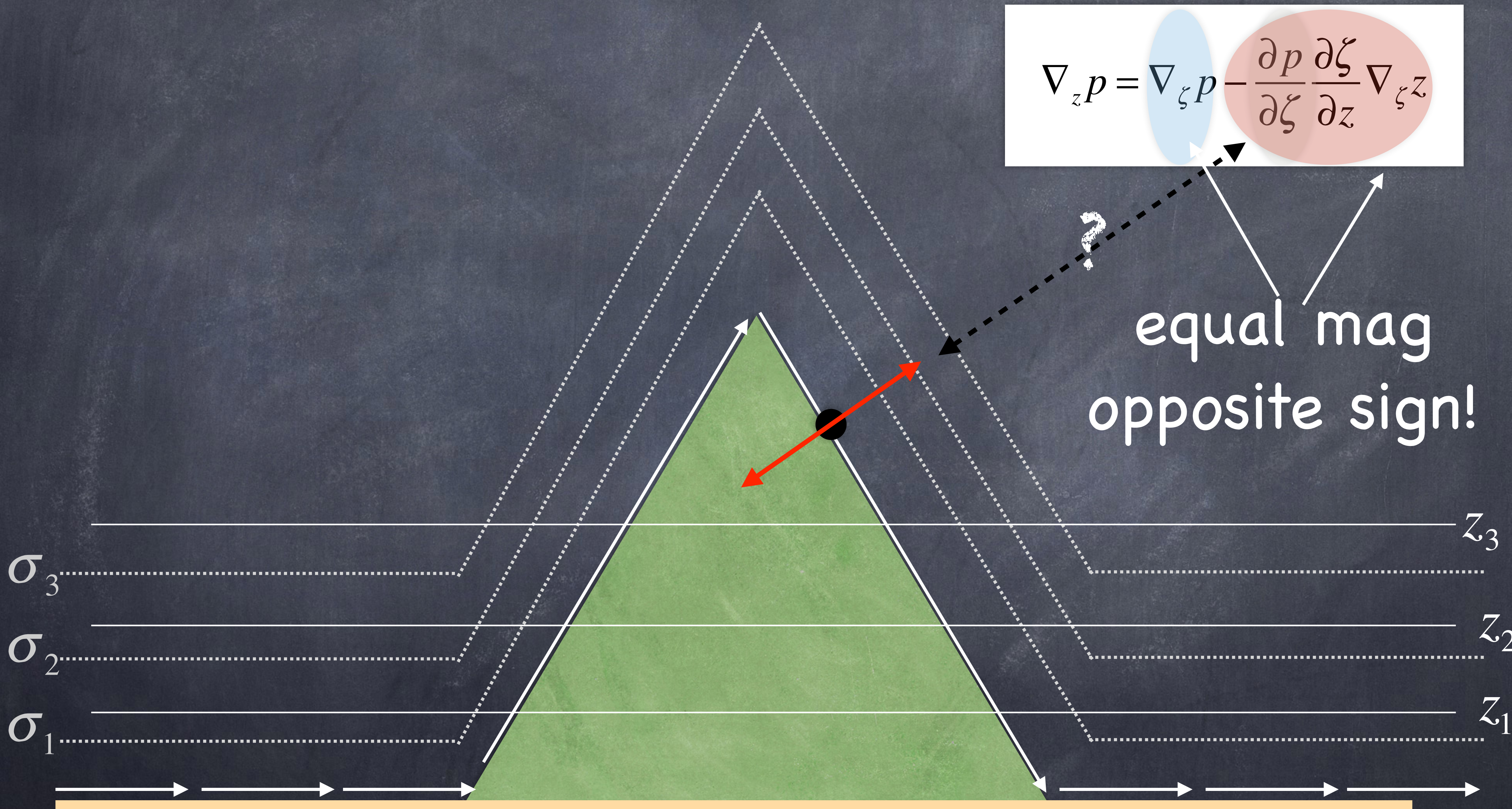
- All horizontal differences are transformed!
- U & V are NOT NOT transformed!
- Therefore PGF for U & V must be "transformed back" -> not in the plane of the transformed coordinate....
- results in PGF corr. term error especially near ground.

PGF Errors

$$\nabla_z p = \nabla_\zeta p - \frac{\partial p}{\partial \zeta}\frac{\partial \zeta}{\partial z}\nabla_\zeta z$$



PGF Errors



$\nabla_z p = \nabla_\zeta p - \frac{\partial p}{\partial \zeta}\frac{\partial \zeta}{\partial z}\nabla_\zeta z$
?
equal mag
opposite sign!
z_3
z_2
z_1
σ_3
σ_2
σ_1

ETA Coordinate

Eta as a Vertical Coordinate

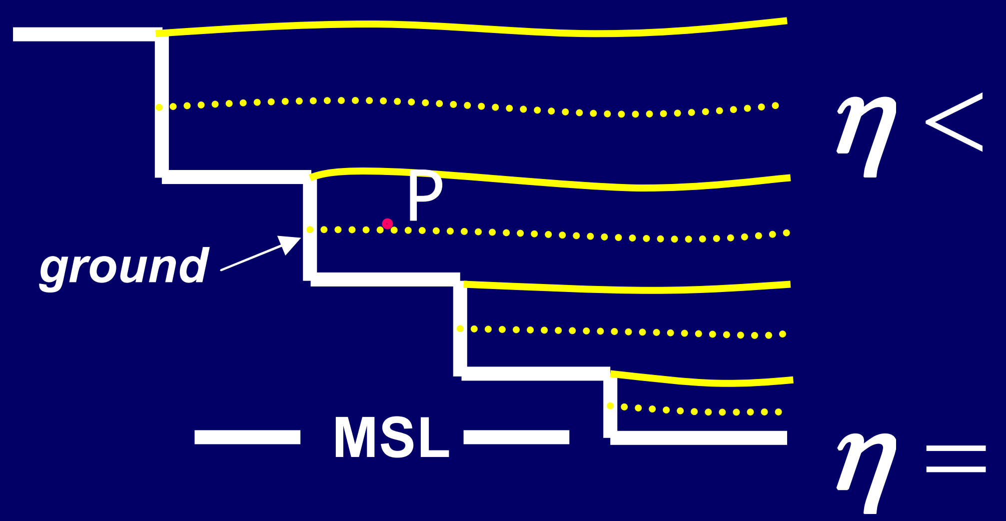
Eta is also called the stepped mountain coordinate. No holes in topography. Tries to reduce the PGF errors using sigma.

Advantage – improves calculation of horizontal pressure gradient force. Performs much better in regions of strong terrain influences

Disadvantage – does not accurately represent the surface topography.

(example NAM 218)

Hybrid pressure/sigma system



$$\eta_s = \frac{p_r(z_s) - p_t}{p_r(z=0) - p_t}$$

- pr(zs) is the pressure in the standard atmosphere at height **zs**
- pt is the pressure at the top of the atmosphere
- pr(z=0) is the pressure at sea level in the standard atmosphere

Shaved Cell Coordinate

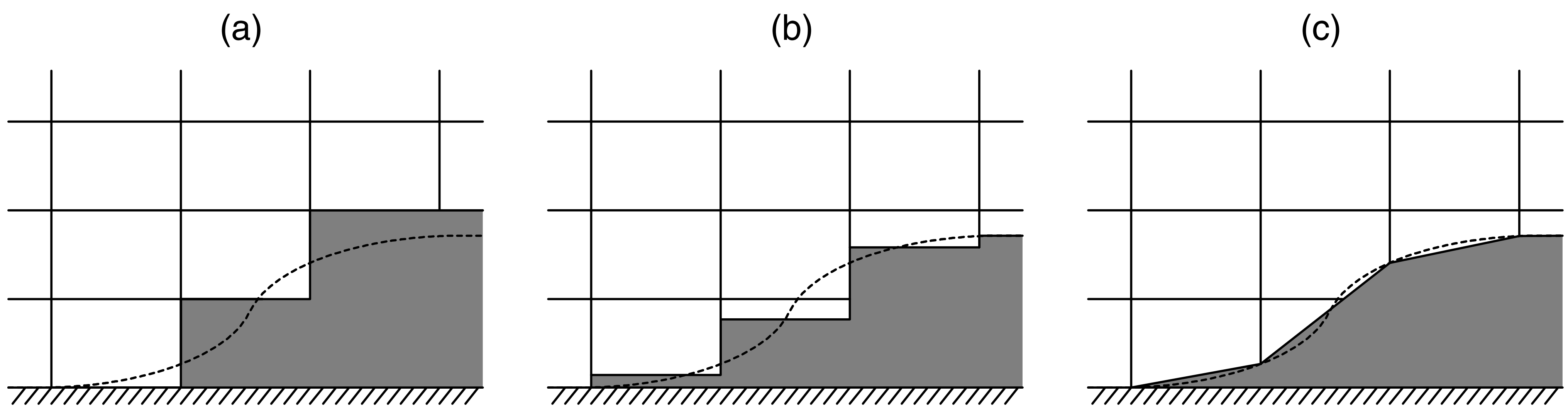


Figure 1. Three z-coordinate topography representations: (a) a box cell method, (b) a partial cell method, and (c) a shave method. Solid lines and dashed lines describe the coordinates and real topography, respectively. Shaded regions describ topographic representations in each model.

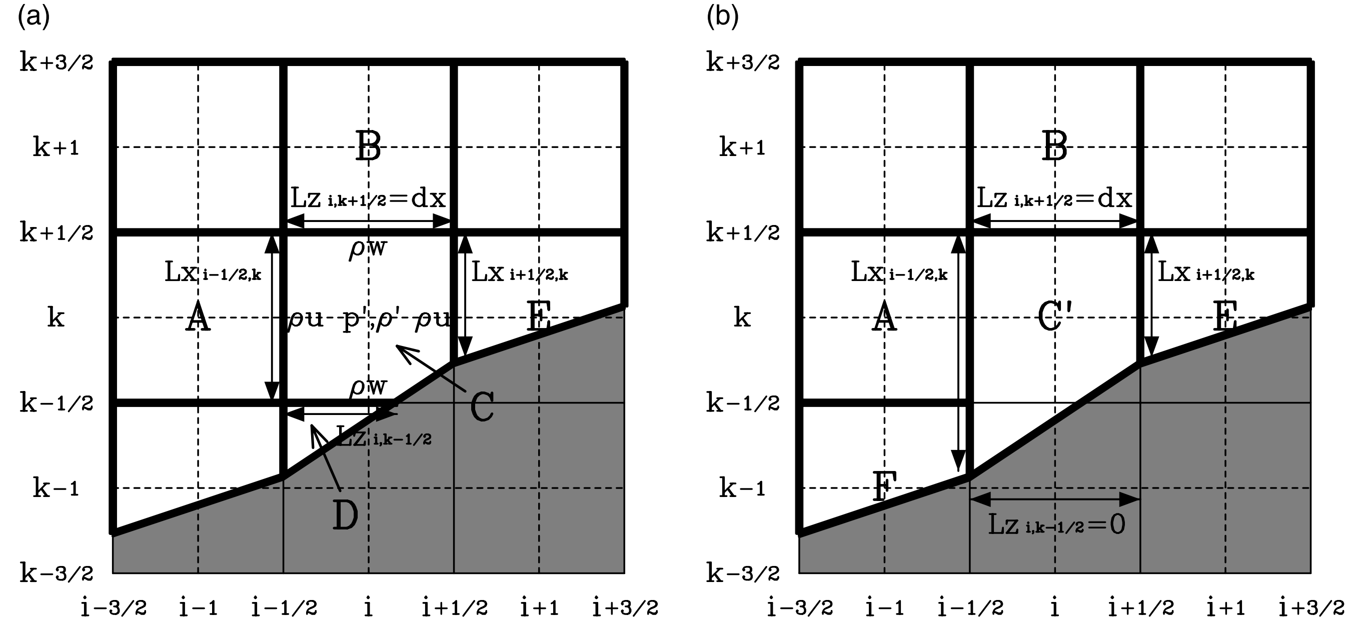
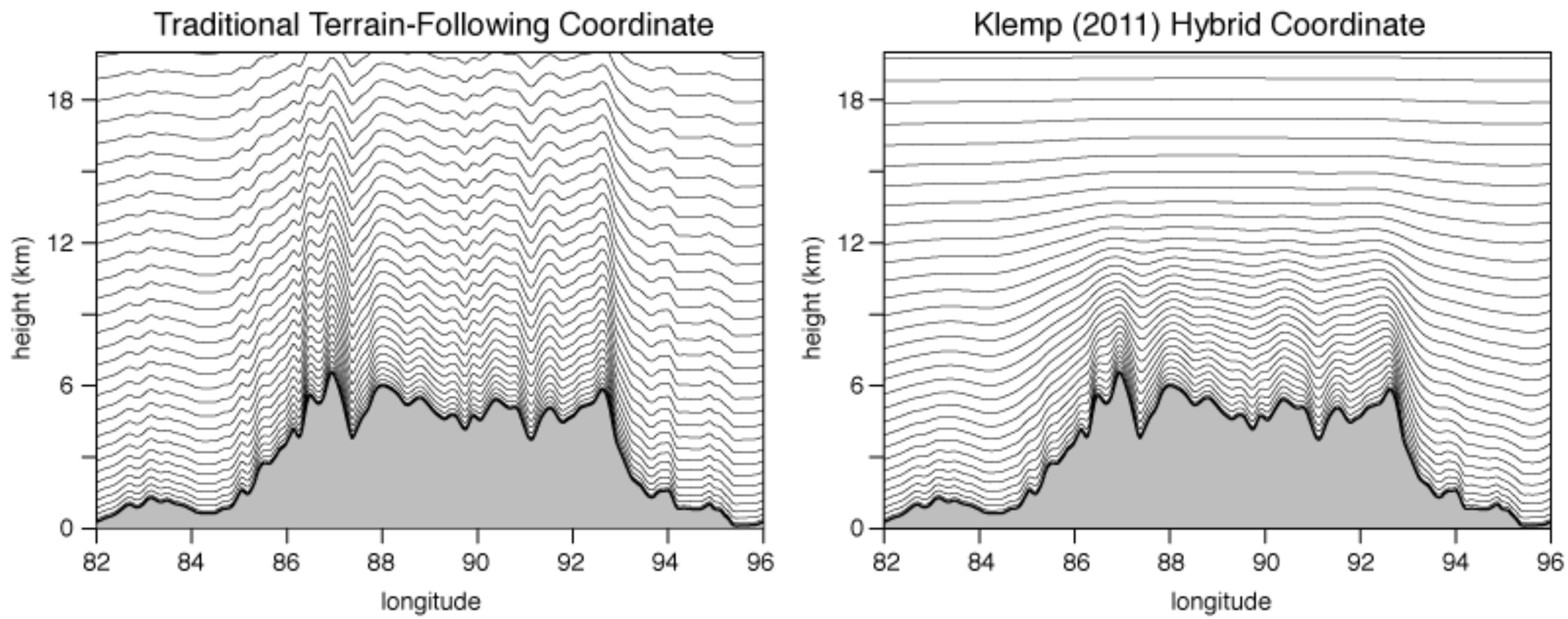


Figure 2. Combination of small cells. Thick lines describe the boundaries of the scalar cells. Shaded regions represent topography in the model. (a) Scalar cells before combination. Scalar cell C exchanges flux with the cells, A, B, D, and E. (b) Scalar cells after combining cells C and D. Combined cell C′ exchanges flux with cells A, B, E, and F.

Sleve Coords



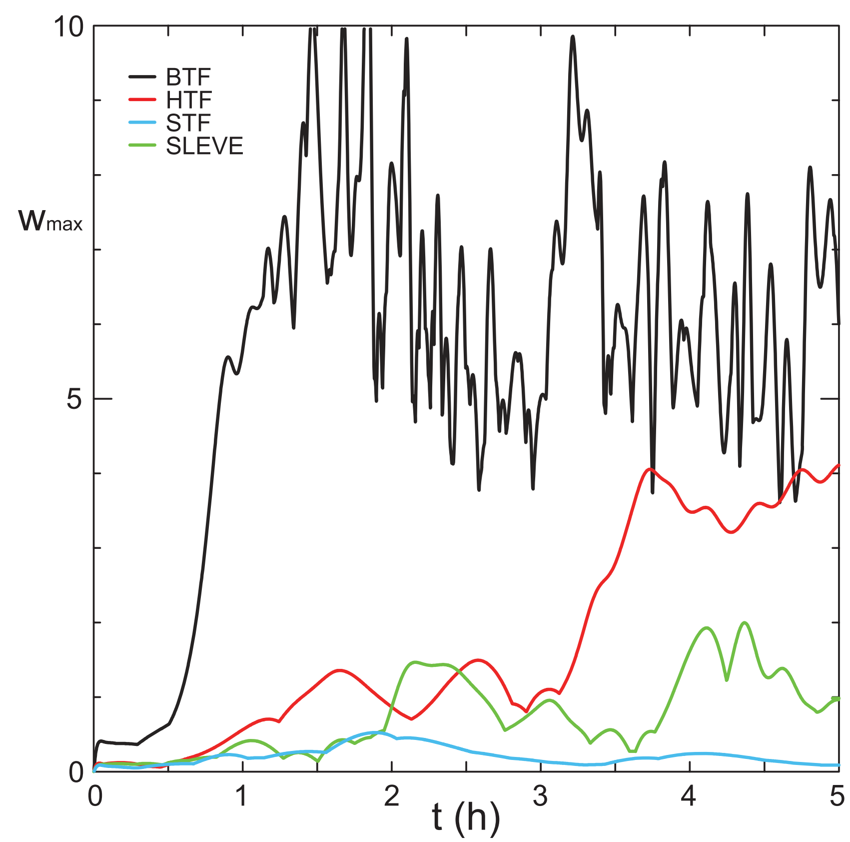
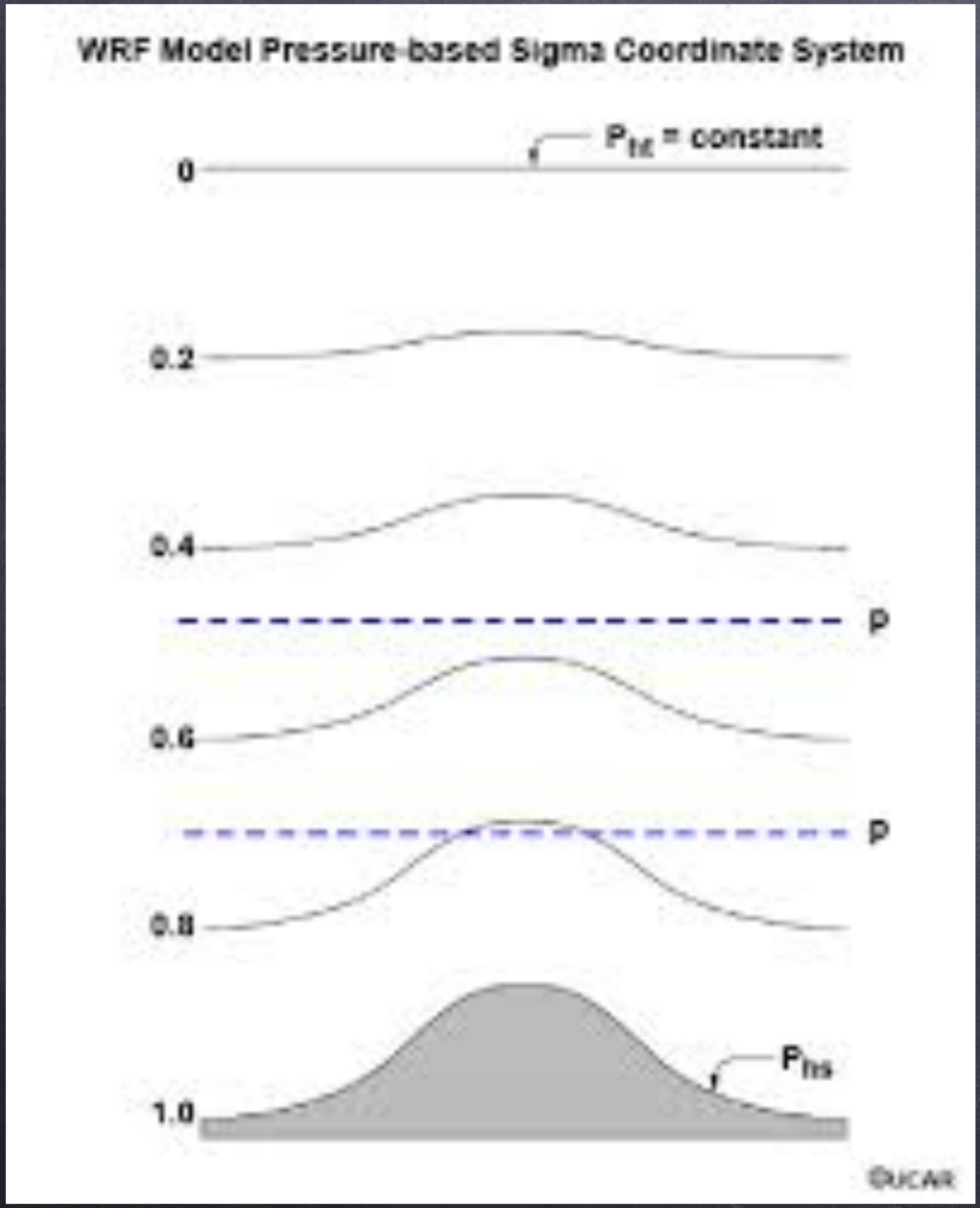


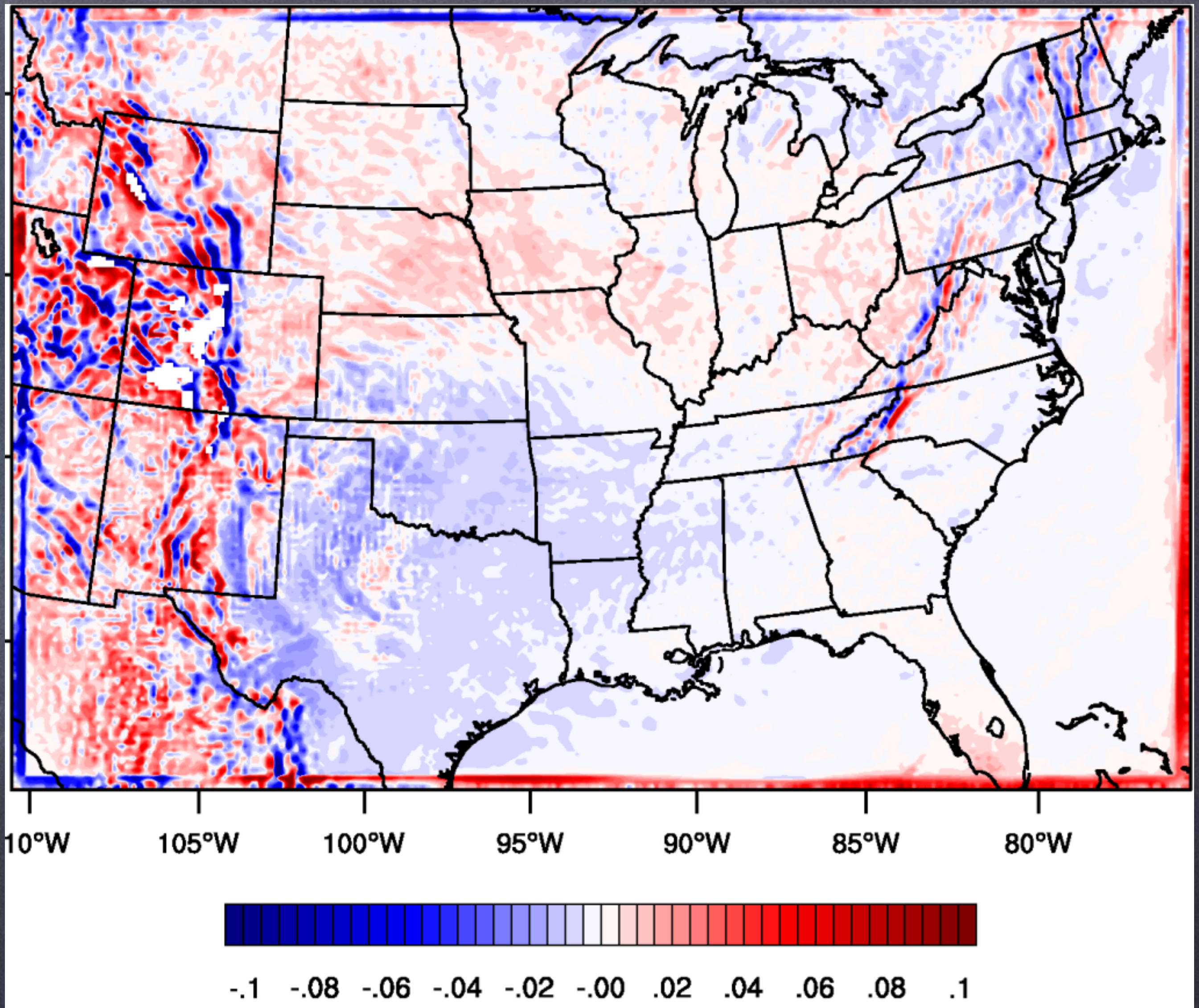
FIG. 2. Time series of the maximum vertical velocity for the

WRF Vertical Coordinate

- Based on Laprise (1992)
- non-hydrostatic pressure coordinate
- coordinate surfaces “move” up and down locally in response to changes in thickness of layers
- Nice in theory - pain in practice
- MPAS (Skamarock’s next model) is Sigma-Z coordinate
- Easier for ensemble data assimilation.....

WRF Vertical Coordinate





End Lecture 2

Begin Lecture 3

Numerical Methods

Representing PDEs

An example of from momentum equation:
U-wind accelerated by only the pressure gradient force........

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - w\frac{\partial u}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

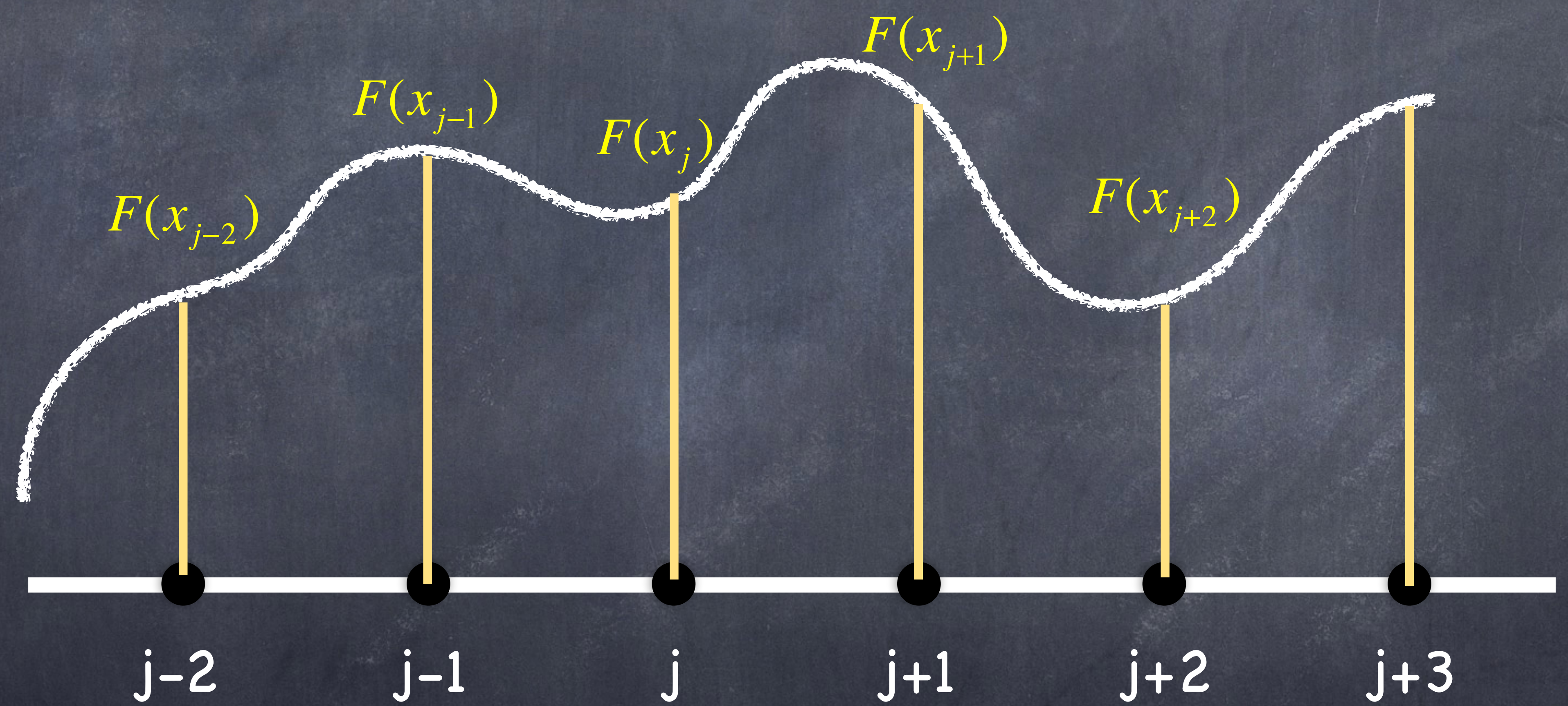
How do you represent these on a computer?

Computers do arithmetic... NOT Calculus!

- Numerical methods
  - represents the continuous with discrete approximations
  - vector calculus
  - integration
  - interpolation
- Goal: convert spatial and temporal derivatives into algebraic equations that computers can solve using addition, subtraction, multiplication, and division (and a few others operations)
- Classes of numerical methods
  - Finite difference and finite volume
    - basis functions are Taylor series
  - Spectral and Galerkin methods (finite element, DG, SE)
    - based on fourier series or local polynomials

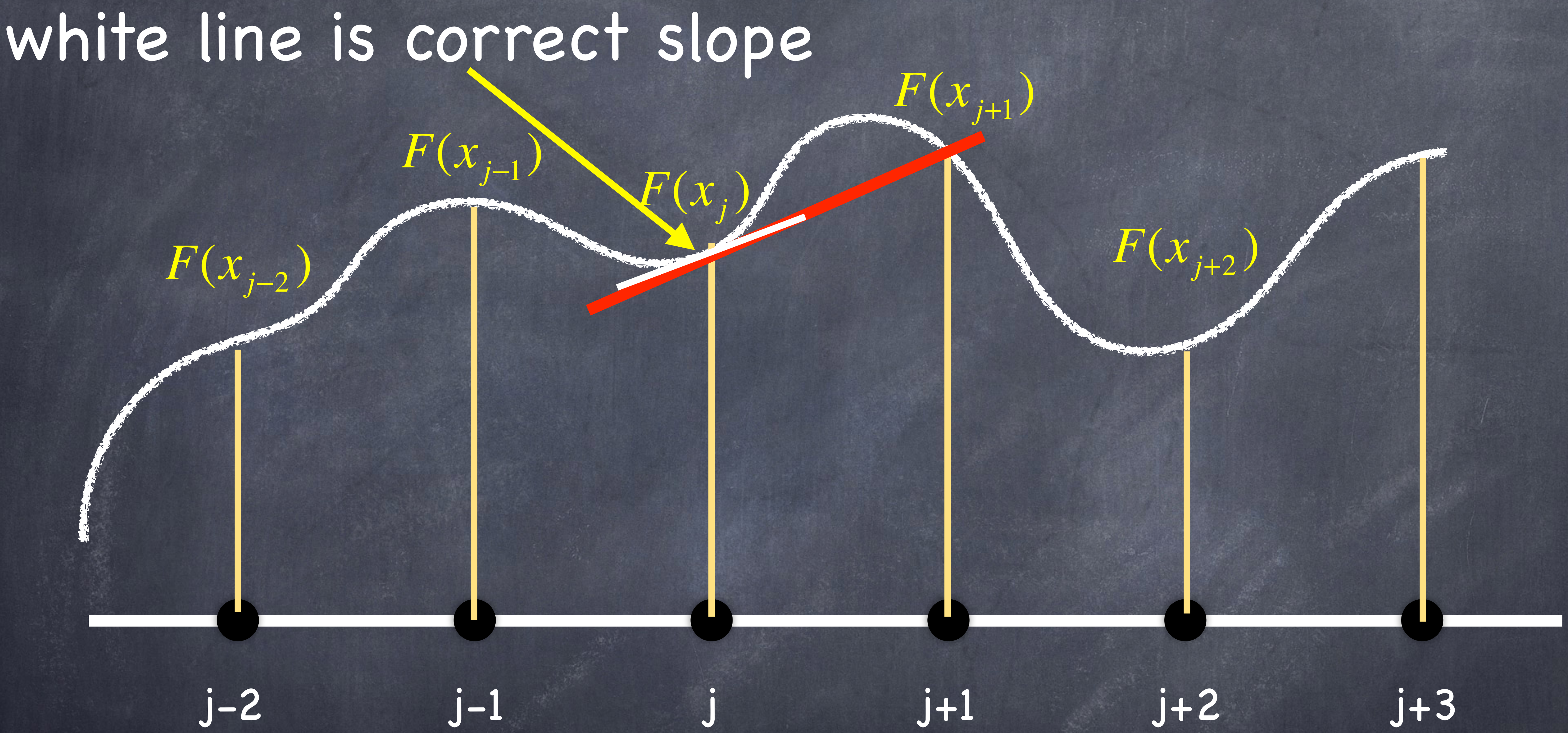
For now, lets focus on Finite Difference approximations (closely related to FV methods)

For now, lets focus on Finite Difference approximations (closely related to FV methods)



F(x)'s represent a set of discrete values

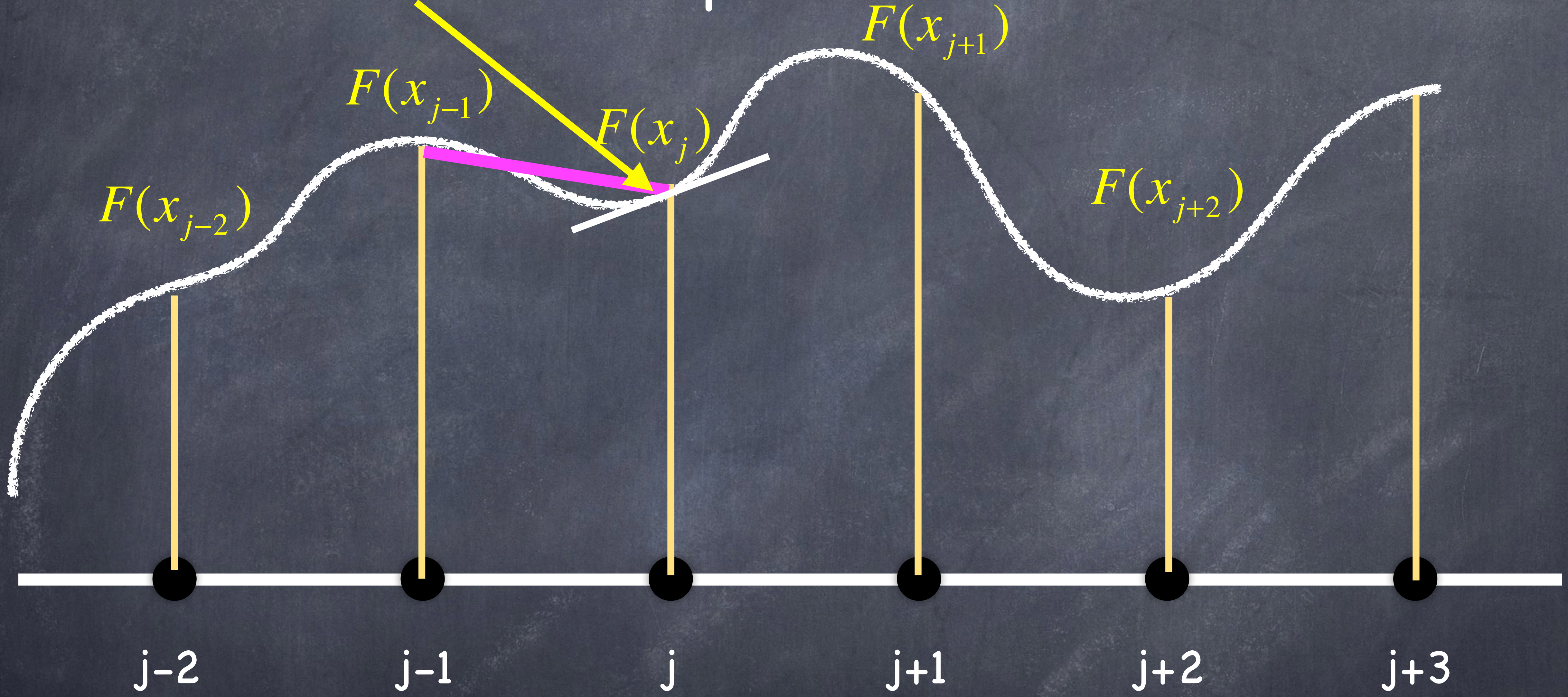
How to take a derivative?



forward difference: $$\left.\frac{\partial F}{\partial x}\right|_{x_j} = \frac{F(x_{j+1}) - F(x_j)}{\Delta x}$$

How to take a derivative?

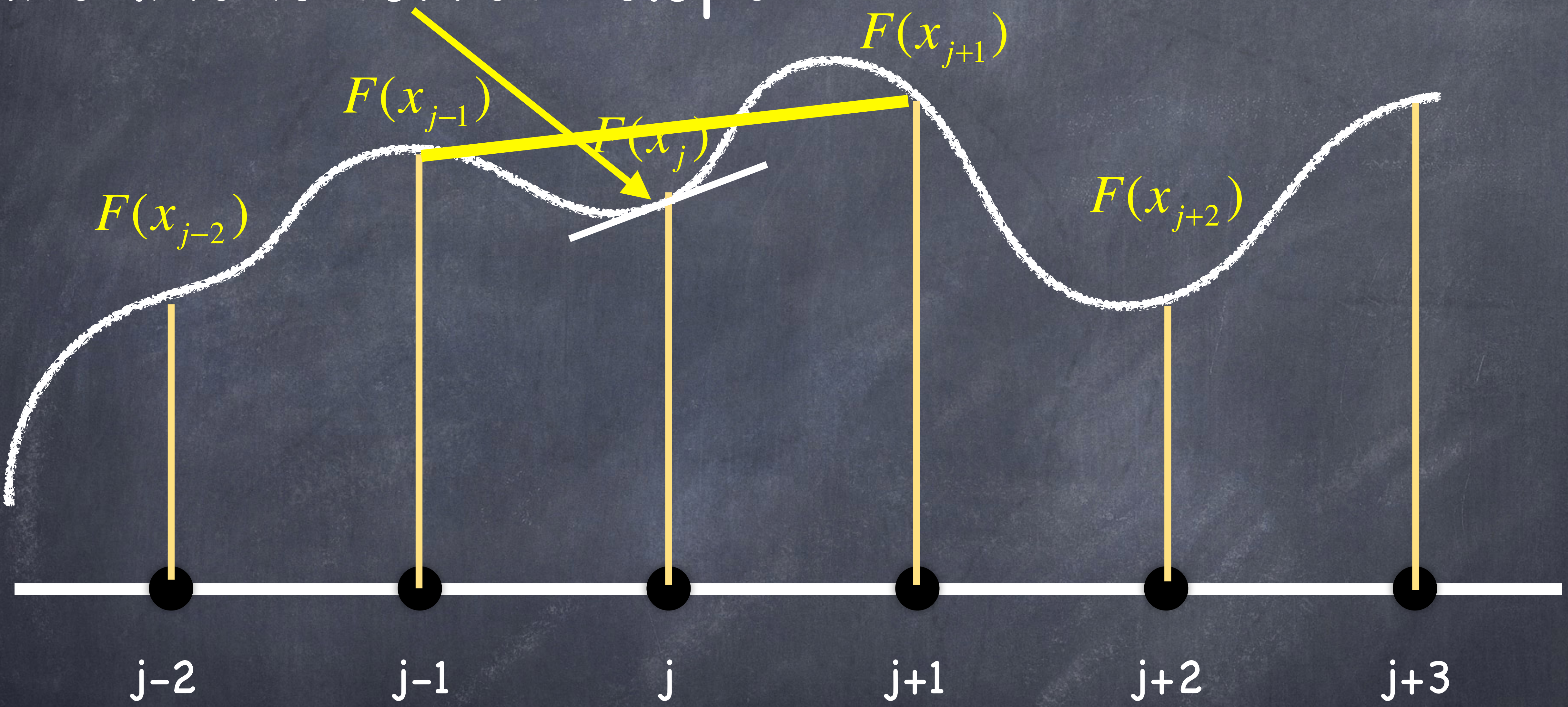
white line is correct slope



backward difference: $$\left.\frac{\partial F}{\partial x}\right|_{x_j} = \frac{F(x_j) - F(x_{j-1})}{\Delta x}$$

How to take a derivative?

white line is correct slope



centered difference: $$\left.\frac{\partial F}{\partial x}\right|_{x_j} = \frac{F(x_{j+1}) - F(x_{j-1})}{2\Delta x}$$

How do we know what is the best approximation?

- In some sense - we don't, but we can perform a Talyor series analysis to estimate the errors associated with a particular scheme
- Talyor series analysis is also used to determine whether as dx ~ 0, does the discrete FDA approximate the PDE (sometimes it does not if you are not careful).
- It is also used, as a first step with a stability analysis method, to determine what the largest time step is permissible for the discrete version of the PDE.

Example: Finite Differences

How to do calculus on a computer?

$$f(x \pm \Delta x) = f(x) \pm \Delta x \left.\frac{\partial f}{\partial x}\right|_x + \frac{\Delta x^2}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_x \pm \ldots + \frac{\Delta x^n}{n!}\left.\frac{\partial^n f}{\partial x^n}\right|_x$$

Classic Taylor series expansion about “x”

To create a derivative...

$$f(x+\Delta x) - f(x-\Delta x) = 2\Delta x \left.\frac{\partial f}{\partial x}\right|_x + \frac{2\Delta x^2}{2!}\left.\frac{\partial^3 f}{\partial x^3}\right|_x + \ldots + \frac{\Delta x^{2(n+1)}}{(n+1)!}\left.\frac{\partial^{2(n+1)} f}{\partial x^{2(n+1)}}\right|_x$$

rearranging...

$$\left.\frac{\partial f}{\partial x}\right|_x = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} = \Delta x^2 \left.\frac{\partial^3 f}{\partial x^3}\right|_x + \ldots + \frac{\Delta x^{2n+1}}{(2n+1)!}\left.\frac{\partial^{2n+1} f}{\partial x^{2n+1}}\right|_x$$

Example: Finite Differences

- What to do with those extra higher-order derivatives?

$$\left.\frac{\partial f}{\partial x}\right|_x = \frac{f(x+\Delta x)-f(x-\Delta x)}{2\Delta x} = \Delta x^2 \left.\frac{\partial^3 f}{\partial x^3}\right|_x + \ldots + \frac{\Delta x^{2n+1}}{(2n+1)!}\left.\frac{\partial^{2n+1} f}{\partial x^{2n+1}}\right|_x$$

- We TRUNCATE! E.g., approximate…here one neglects the terms associated with the third-derivative of function (f).

$$\left(\frac{\partial f}{\partial x}\right)_i = \frac{f(x+\Delta x)-f(x-\Delta x)}{2\Delta x} = \frac{f_{i-1}-f_{i+1}}{2\Delta x} + O\left(\Delta x^2\right)$$

-

Example: Finite Differences

- What to do with those extra higher-order derivatives?

$$\left.\frac{\partial f}{\partial x}\right|_x = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} = \Delta x^2 \left.\frac{\partial^3 f}{\partial x^3}\right|_x + \ldots + \frac{\Delta x^{2n+1}}{(2n+1)!}\left.\frac{\partial^{2n+1} f}{\partial x^{2n+1}}\right|_x$$

- We TRUNCATE! E.g., approximate…here one neglects the terms associated with the third-derivative of function (f).

$$\left(\frac{\partial f}{\partial x}\right)_i = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} = \frac{f_{i-1} - f_{i+1}}{2\Delta x} + O\left(\Delta x^2\right)$$

"2nd order" approximation

- "2nd-order" implies that as dx is reduced, the truncation error reduces quadratically (if F is smooth!). This is called convergence.
- **Truncation is always necessary** (finite difference, spectral, etc).
- Truncation is one of the underlying approximation errors for the underlying PDEs

Derivation of spatial errors

Analytical

$$F(x) = A_n e^{ikx} = e^{ikx}$$

$$\frac{\partial F}{\partial x} = ik\, e^{ikx}$$

Finite Difference

$$F(x_j) = e^{ikx}$$

$$\frac{\partial F}{\partial x} = \left[\frac{e^{ik(x+\Delta x)} - e^{ik(x-\Delta x)}}{2\Delta x}\right]$$

$$= e^{ikx}\left[\frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x}\right]$$

$$\frac{\partial F}{\partial x} = \frac{i\sin(k\Delta x)}{\Delta x} e^{ikx}$$

Useful things

$$k = \frac{2\pi}{L} \qquad k\Delta x = \frac{2\pi\Delta x}{L}$$

$$k\Delta x = 0 \qquad k\Delta x = \frac{\pi}{2} \qquad k\Delta x = \pi$$

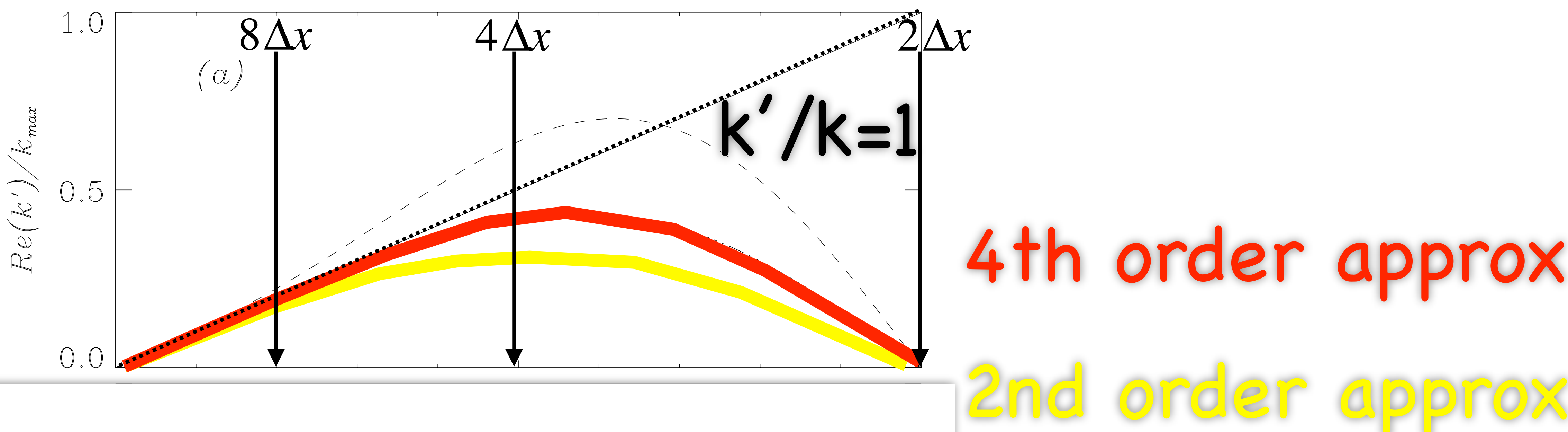
$$L = \infty \qquad L = 4\Delta x \qquad L = 2\Delta x$$

$$\frac{k'}{k} = \frac{\sin(k\Delta x)}{k\Delta x}$$

as kdx -> 0 => 1 (L'Hopital's rule)

as kdx -> 3.14 => 0!

Errors from spatial approx..cont.



When L >> dx: derivatives are accurate
When L = 4dx: large errors
When L = 2 dx: 100% error

What about approximating temporal derivatives?

- Similar procedure as for FDA
- Lots of rich theory due to approximations for ODE's
- Two classes of schemes most often used....
  - Runge Kutta (multistep)
  - Adams methods (multistage)

Atmos NWP

- Leapfrog scheme (MM5, GFS)
- Runge Kutta (WRF, COSMO, NICAM, MPAS)
- Adams (NAM)

LeapFrog Scheme with advection equation

$$\frac{\partial T}{\partial t} = -U\frac{\partial T}{\partial x}$$

$$\left(\frac{\partial T}{\partial t}\right)_n = \frac{T(t+\Delta t) - T(t-\Delta t)}{2\Delta t} = \frac{T^{n+1} - T^{n-1}}{2\Delta t} + O\left(\Delta t^2\right)$$

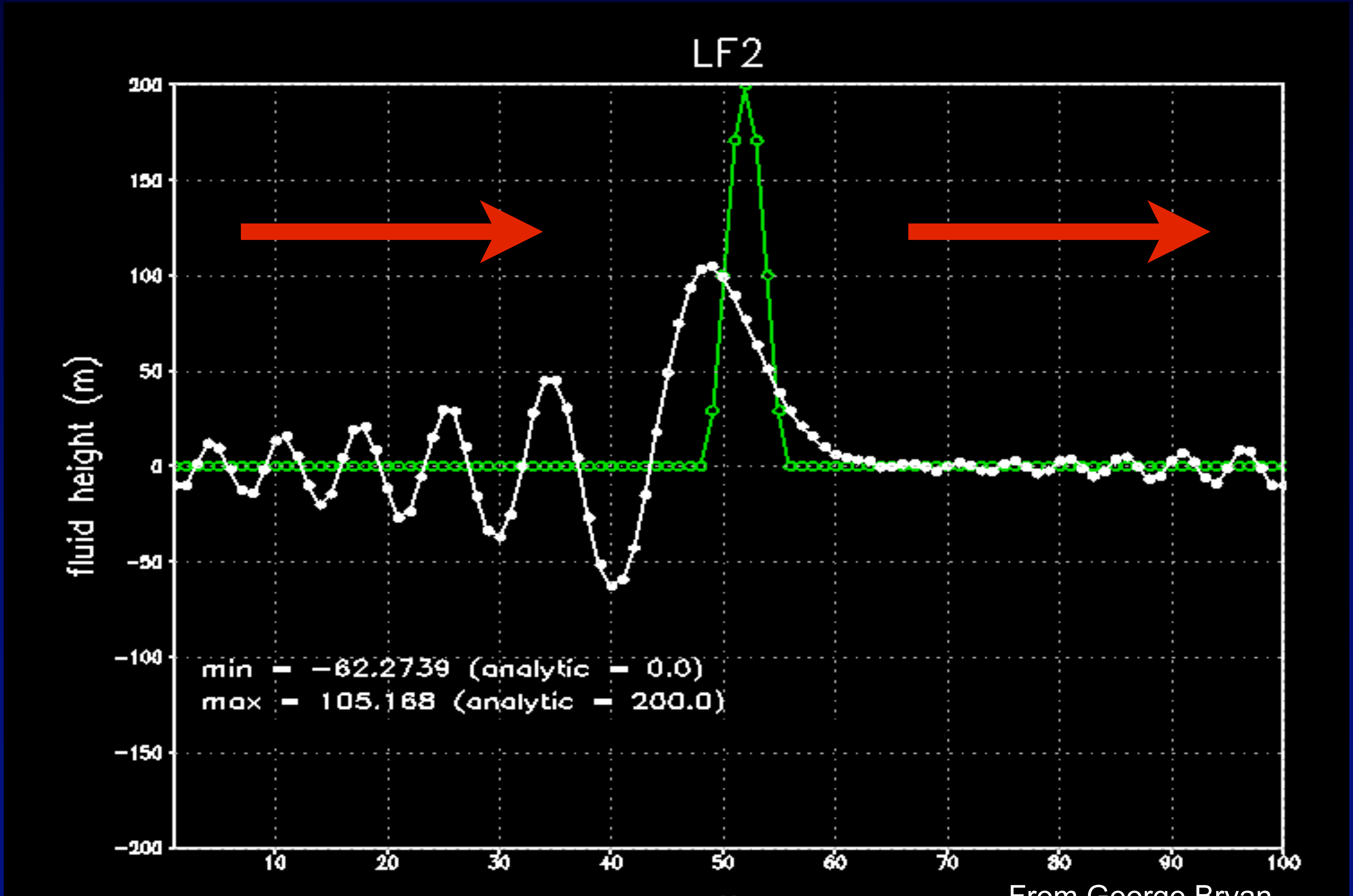
$$\left(\frac{\partial T}{\partial x}\right)_i = \frac{T(x+\Delta x) - T(x-\Delta x)}{2\Delta x} = \frac{T_{i-1} - T_{i+1}}{2\Delta x} + O\left(\Delta x^2\right)$$

$$\frac{T_j^{n+1} - T_j^{n-1}}{2\Delta t} = -U\frac{T_{j+1}^n - T_{j-1}^n}{2\Delta x} \longrightarrow T_j^{n+1} = T_j^{n-1} - \frac{U\Delta t}{\Delta x}\left(T_{j+1}^n - T_{j-1}^n\right)$$

2nd order in time & space

Approximating 1D advection $\frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial x}$

- MM5: leapfrog (t) and 2nd-order centered (x)



From George Bryan

RK3 Scheme with advection equation

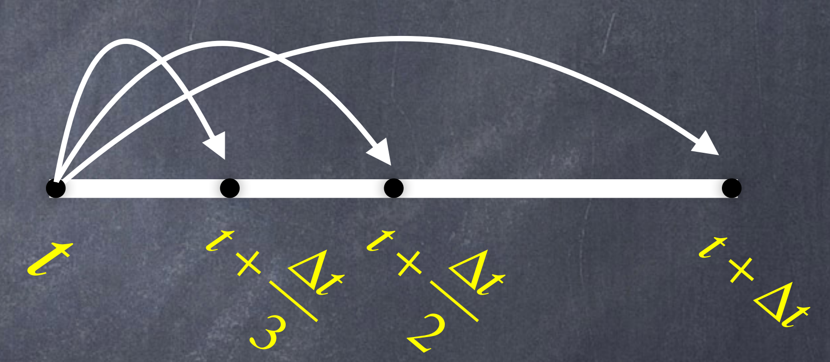
$$\frac{\partial T}{\partial t} = -U\frac{\partial T}{\partial x}$$

RK-> Multistep scheme
do multiple passes
from t to t+dt

$$T_j^1 = T_j^n - \frac{\Delta t}{3}\left[\frac{2}{3}\left(T_{j+1}^n - T_{j-1}^n\right) - \frac{1}{12}\left(T_{j+2}^n - T_{j-2}^n\right)\right]$$

$$T_j^2 = T_j^n - \frac{\Delta t}{2}\left[\frac{2}{3}\left(T_{j+1}^1 - T_{j-1}^1\right) - \frac{1}{12}\left(T_{j+2}^1 - T_{j-2}^1\right)\right]$$

$$T_j^{n+1} = T_j^n - \Delta t\left[\frac{2}{3}\left(T_{j+1}^2 - T_{j-1}^2\right) - \frac{1}{12}\left(T_{j+2}^2 - T_{j-2}^2\right)\right]$$



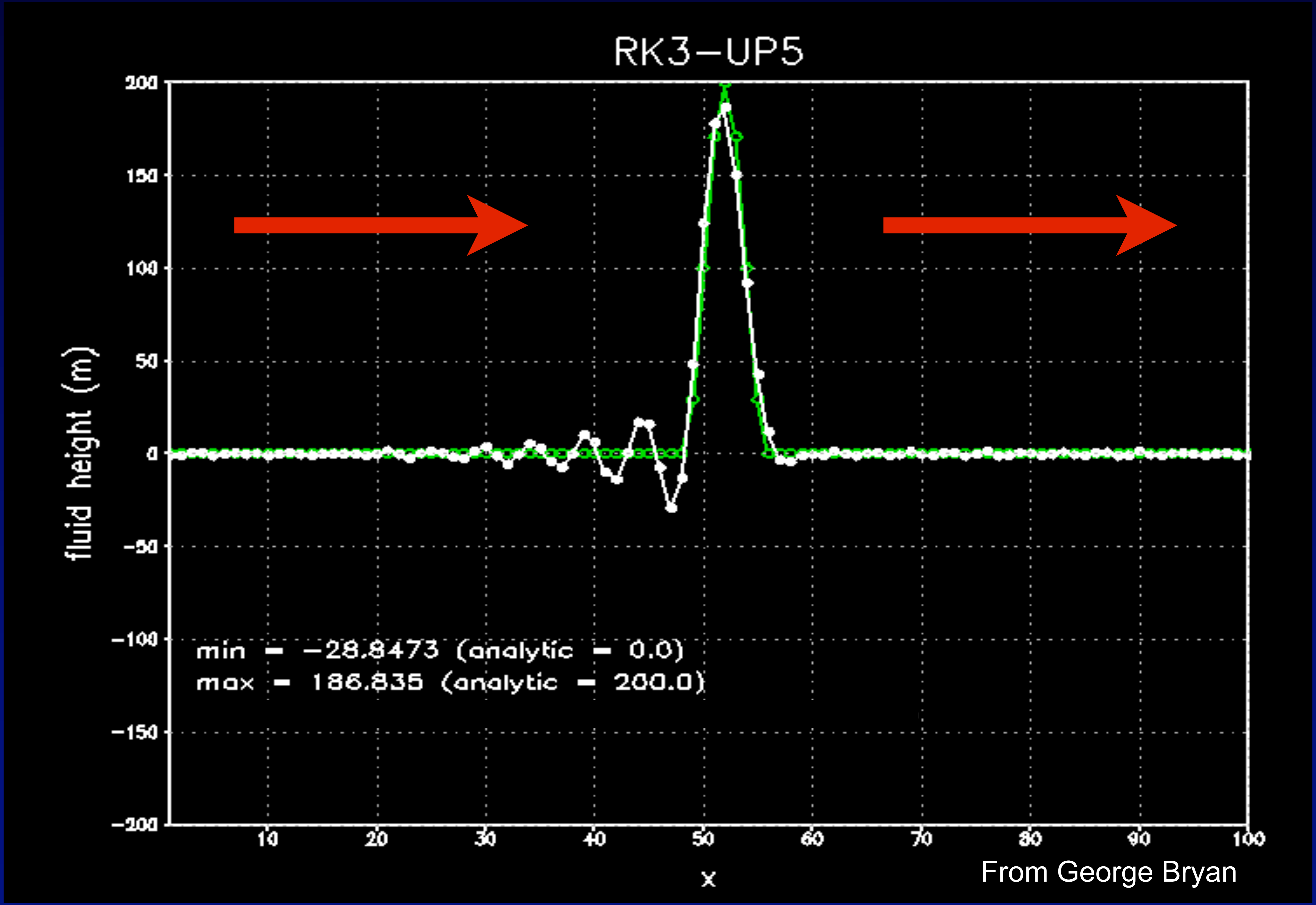
Costs more (3 RHS)
But more accurate

3rd order in time & 4th order space

Approximating 1D advection

$$\frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial x}$$

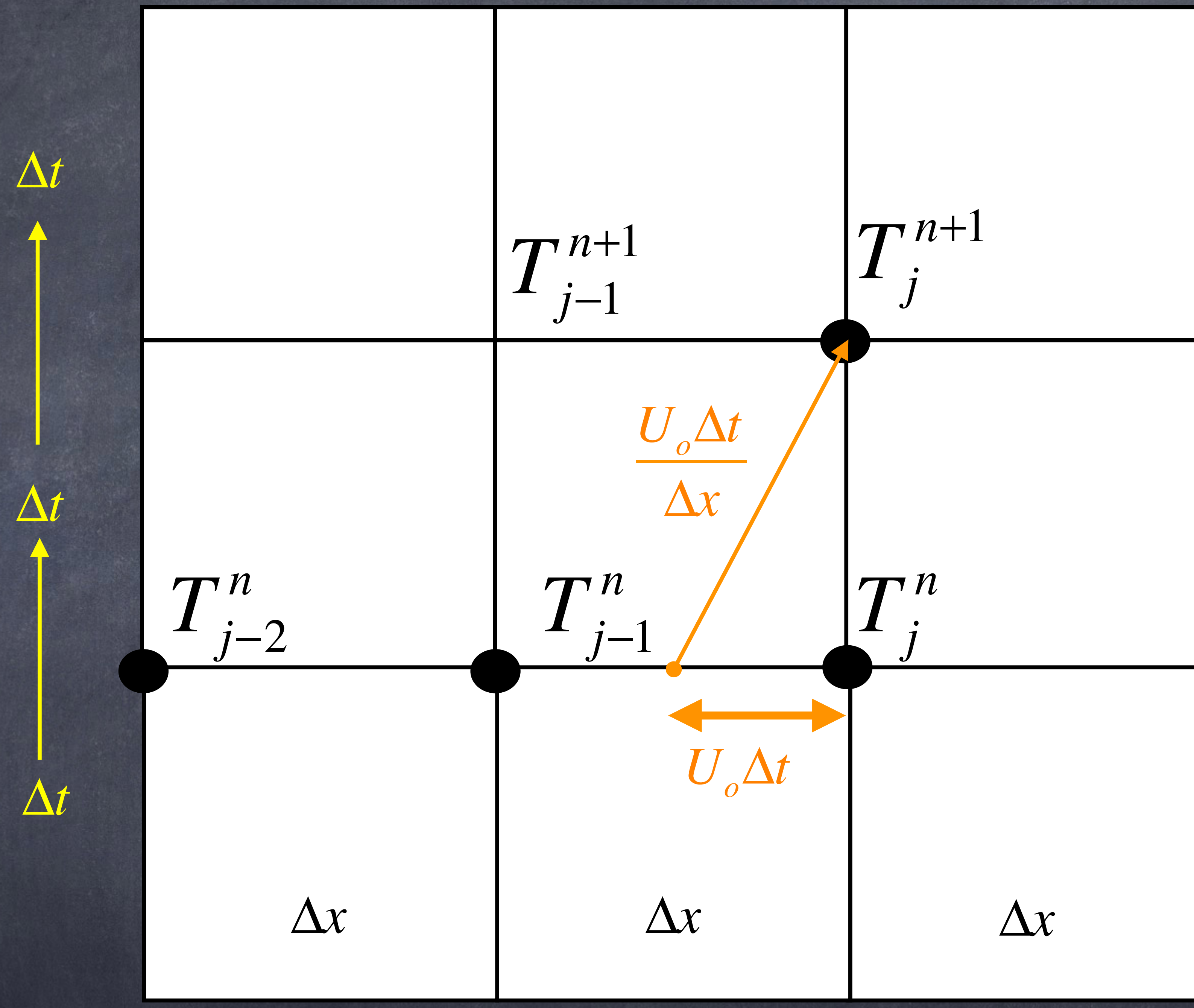
- WRF: Runge-Kutta (t) and 6th-order centered (x)



Stability of numerical schemes....

- Numerical schemes need to have the same behavior as they PDEs they represent.
- For hyperbolic PDES - stuff or waves should propagate at approximately right speeds
- For parabolic PDEs (diffusion), the highest wave numbers in the initial solution should dampen the quickest.
- We have talked about about truncation error
- Lets talk about stability of FDAs

Example: Upwind scheme



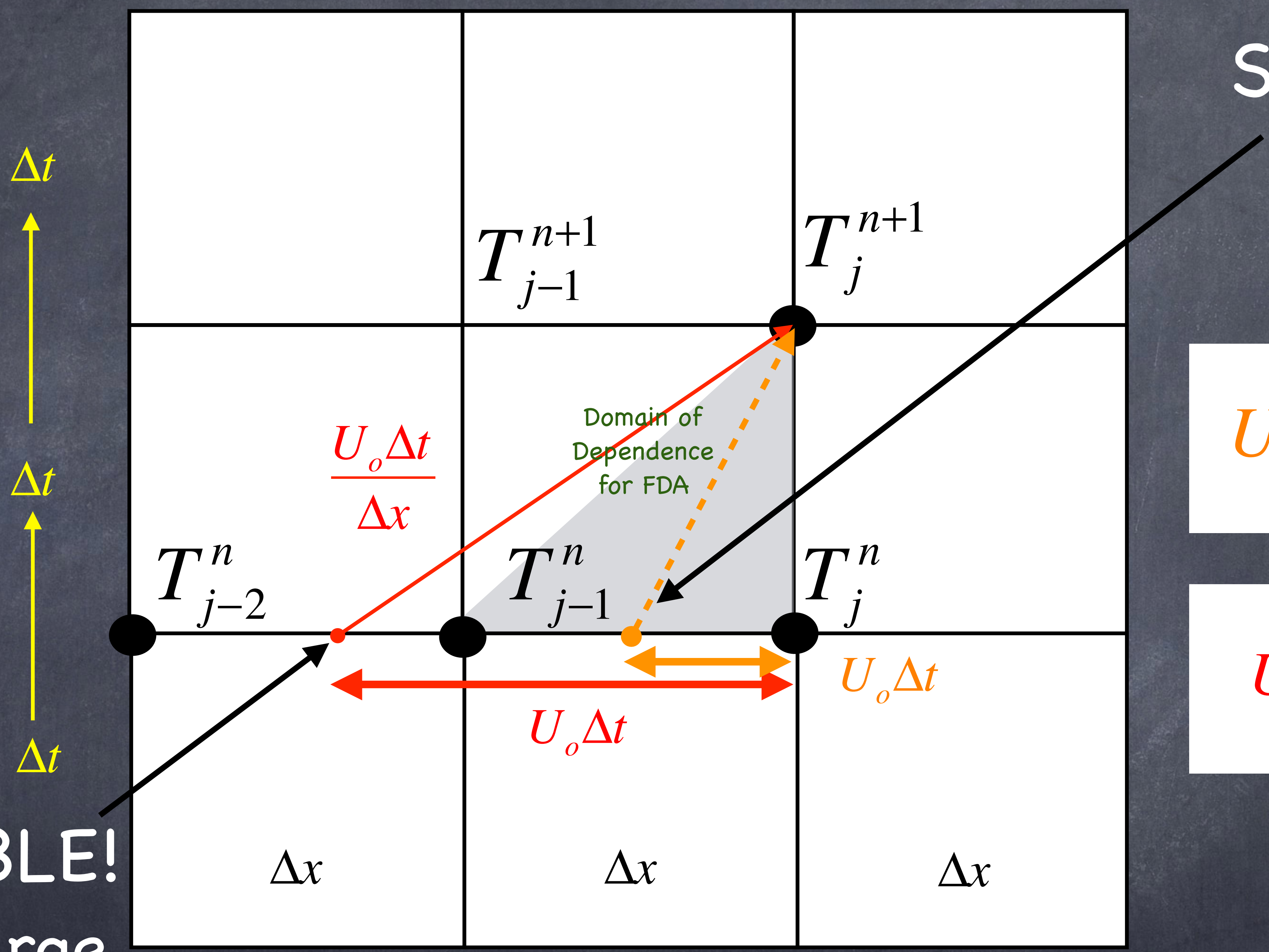
$$U_o \Delta t \le \Delta x$$

Stable!

$$\frac{T_j^{n+1} - T_j^n}{\Delta t} = -U_o \frac{T_j^n - T_{j-1}^n}{\Delta x} \dashrightarrow T_j^{n+1} = T_j^n - \frac{U_o \Delta t}{\Delta x}\left(T_j^n - T_{j-1}^n\right)$$

Example: Upwind scheme

STABLE!



$$U_o \Delta t \le \Delta x$$

$$U_1 \Delta t \ge \Delta x$$

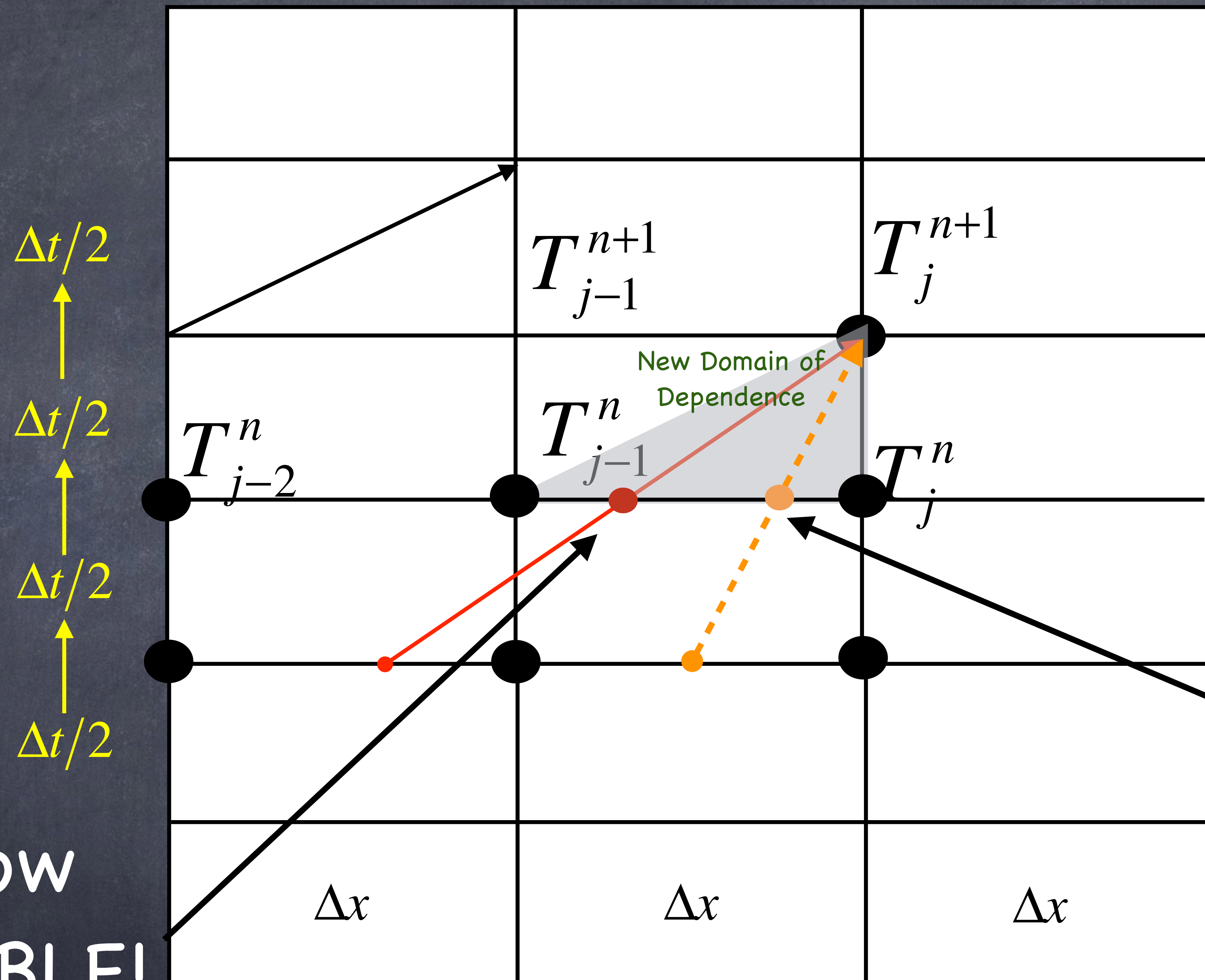
UNSTABLE!
dt too large

$$\frac{U \Delta t}{\Delta x} < 1$$

Needed for Stability!

CFL condition

Upwind scheme



$$U_o \frac{\Delta t}{2} \le \Delta x$$

$$U_1 \frac{\Delta t}{2} \le \Delta x$$

Still STABLE!

Now STABLE!

$$\frac{U \Delta t}{\Delta x} < 1$$

Can we derive this formally?

Yes - using von Neuman analysis

Stability analysis

$$\frac{\partial f}{\partial t} = -U\frac{\partial f}{\partial x}$$

$$f_j^n = A_n e^{ikj\Delta x}$$

Advection: IC moves, but does not grow or damp!

$$\left|\frac{f_j^{n+1}}{f_j^n}\right| = \left|\frac{A_{n+1}e^{ikj\Delta x}}{A_n e^{ikj\Delta x}}\right| = \left|\frac{A_{n+1}}{A_n}\right| = |\lambda|$$

$|\lambda|$ Is amplification factor

Stability analysis

$|\lambda| < 1$

$|\lambda| = 1$

$|\lambda| > 1$

For advection eq, which is the correct behavior?

We can derive what lamba is by plugging in the discrete waveform for each approximation and derive under what conditions lamba <= 1

Upstream scheme stability analysis

$$\frac{\partial f}{\partial t} = -U\frac{\partial f}{\partial x} \qquad f_j^n = A_n e^{ikj\Delta x}$$

$$f_j^{n+1} = f_j^n - \frac{U\Delta t}{\Delta x}\left(f_j^n - f_{j-1}^n\right)$$

$$A^{n+1}e^{ikj\Delta x} = A^n e^{ikj\Delta x} - \frac{U\Delta t}{\Delta x}A^n e^{ikj\Delta x}\left(1 - e^{-ik\Delta x}\right)$$

$$\frac{A^{n+1}}{A^n} = 1 - \frac{U\Delta t}{\Delta x}\left(1 - e^{-ik\Delta x}\right) = 1 - c_r\left(1 - e^{-ik\Delta x}\right)$$

$$\left|\frac{A^{n+1}}{A^n}\right| = \left|1 - c_r\left(1 - e^{-ik\Delta x}\right)\right| = |\lambda|$$

Now have an expression for the amplification factor!

Upstream scheme stability analysis

$$|\lambda| = \left|1 - c_r\left(1 - e^{-ik\Delta x}\right)\right|$$

$$|\lambda| = \left|1 - c_r\left(1 - \cos(k\Delta x) + i\sin(k\Delta x)\right)\right|$$

Time for some trig!

$$|\lambda| = \left(\mathrm{Re}^2 + \mathrm{Im}^2\right)^{1/2} = \left(\left(1 - cr + cr\cos(k\Delta x)\right)^2 + \sin^2(k\Delta x)\right)^{1/2}$$

$$|\lambda| = \left(1 - 2c_r\left(1 - \cos(k\Delta x)\right)(1 - c_r)\right)^{1/2}$$

magical trig.....

Using this expression, we check conditions which keep $|\lambda| \leq 1$. Run through the cases:

(a) if $c_r < 0$, then $|\lambda| > 1$. The "downwind" scheme is absolutely unstable.

(b) if $c_r > 1$, then $|\lambda| > 1$ if $(1 - \cos k\Delta x) > 0$, which is is always the case. Absolutely unstable again.

(c) if $c_r \leq 1$, then need to check the cosine term. Run through the $k\Delta x$ possibilities.

if $k\Delta x = 0$,	$1 - \cos k\Delta x = 0$,	then $|\lambda| = 1$
if $k\Delta x = \pi/2$,	$1 - \cos k\Delta x = 1$,	then $|\lambda| = (1 - 2c_r(1 - c_r))^{1/2}$
if $k\Delta x = \pi$,	$1 - \cos k\Delta x = 2$,	then $|\lambda| = (1 - 4c_r(1 - c_r))^{1/2}$

As long as $0 < c_r \leq 1$, then $|\lambda| \leq 1$. This is called conditional stability, or the CFL condition. Almost all schemes have this type of stability restriction.

Stability Analysis: Amplitude Errors....

$$|\lambda| = \left(1 - 2c_r\left(1 - \cos(k\Delta x)\right)\left(1 - c_r\right)\right)^{1/2}$$

Note that the amplitude of the scheme for some cases is much less than 1. Just as in the growth case, we can get expodential decay of the solution as well. Let $k\Delta x = \pi/4$ and $c_r = 0.5$. What is the amplitude after 10, 20, and 100 time steps?

$$|\lambda| = \left(1 - 2 \times \tfrac{1}{2}\left(1 - \cos\tfrac{\pi}{4}\right)\left(1 - \tfrac{1}{2}\right)\right)^{1/2} = .924$$

$$|\lambda|^{10} = .45$$

$$|\lambda|^{20} = .20$$

$$|\lambda|^{100} \sim 0$$

Therefore for an $8\Delta x$ wave, no amplitude remain after 100 time steps.

Stability Analysis: Phase Errors....

characteristic!

$$\lambda = 1 - c_r\left(1 - \cos(k\Delta x) + i\sin(k\Delta x)\right)$$

Phase Errors

We can also use the stability analysis to tell us something about the phase speed of the each wave component. What should the phase speed be (it should be 'c'). Lets first find the phase speed for the analytical solution.

$$f(x,t) = F(x-ct) = F_o e^{ik(x-ct)} \quad where\ f(x,0) = F_o e^{ikx}$$

for a single plane wave

Compare the solutions at sucessive time steps

$$\frac{f(x,t+\Delta t)}{f(x,t)} = \frac{F_o e^{ik(x-c(t+\Delta t))}}{F_o e^{ik(x-ct)}} = e^{-ikc\Delta t}$$

c = U

Stability Analysis: Phase Errors….

$$\lambda = 1 - c_r\left(1 - \cos(k\Delta x) + i\sin(k\Delta x)\right)$$

Now the change in phase of the wave per time step will be given by θ_a

$$\theta_a = \tan^{-1}\left(\frac{\mathrm{Im}}{\mathrm{Re}}\right) = \tan^{-1}\left(\frac{\sin kc\Delta t}{\cos kc\Delta t}\right) = -kc\Delta t$$

This is the analytical phase of the solution. Now the phase speed of a wave is given by ω/k. In the expression above,

$$kc\Delta t = \frac{2\pi}{L}\frac{L}{T}\Delta t = \omega_a \Delta t \qquad define\ \ \omega_a = kc$$

θ_a describes the change of phase per time step. Since ω = kc, note that ω/k = c, which is exactly the phase speed you should get! Notice that the phase speed ω/k is NOT a function of k, that means that all the waves travel at the same speed. This is a non-dispersive solution.

Stability Analysis: Phase Errors….

In a similar way, we can find the numerical phase of the upstream scheme, from the

$$\theta_n = \tan^{-1}\left(\frac{-c_r \sin k\Delta x}{1 - c_r(1 - \cos k\Delta x)}\right) = \omega_n \Delta t$$

Here, ω_n/k IS a function of k -> therefore each wavelength travels at a different speed. The solution is dispersive!

Now to examine how the waves move, we take the ratio of the analytical phases to the numerical phases.

$\frac{\theta_n}{\theta_n} < 1$ waves move slower than c

$\frac{\theta_n}{\theta_n} = 1$ waves speeds are exact

$\frac{\theta_n}{\theta_n} > 1$ waves move faster than c

obviously, the best ratio is where the numerical phase speeds match the analytical phase speeds. For the upstream scheme, we get

$$\frac{\theta_n}{\theta_a} = \frac{\tan^{-1}\left(\frac{-c_r \sin k\Delta x}{1 - c_r(1 - \cos k\Delta x)}\right)}{-kc\Delta t}$$

which does not tell you much, so use a computer to plot the ratio. You plot the phase speed as a function of c_r and $k\Delta x$.

Summary for Approximations

- *Numerical methods do really matter!*
  - approximation errors are largest when features are smallest
  - approximations with higher-order truncation (e.g., 6th versus 2nd) have lower phase and amplitude errors for linear advection.
  - How you approximate the temporal derivatives is also important for motions....
- "Effective resolutions" for spatial finite differences approximations......
  - 2nd order FDAs: features < 16 dx are poorly represented
  - 4th order FDAs: features < 10 dx are poorly represented
  - 6th order FDAs: features < 6-7 dx are poorly represented
- Spectral models are much more accurate per "dx", but also cost much more than finite differences. BC's can be more complicated, especially if model is limited area.
- Nearly all original limited area NWP models used 2nd order approximations - despite the limits of that approximation - they still made useful predictions.
- Numerics is only part of the story - PHYSICS is also important to NWP!

End Lecture 3

Numerical Methods

Begin Lecture 4

Parameterizations

Parameterizations

Parameterizations approximate the bulk effects of physical processes that are too small, too complex, or too poorly understood to be explicitly represented in the set of algebraic equations that integrate the PDEs

What do we mean by “Physics"

- Physics: Two “categories”
  - Inputs of momentum, heat and moisture from the boundaries of the domain (earth and space)
    - friction
    - sea surface fluxes
    - solar radiation
  - processes that are too small to be resolved on a numerical grid
    - ice nucleation on CCN
    - melting of graupel into rain
    - vertical transport of heat, momentum and moisture from convective plumes in the boundary layer
- Both require PARAMETERIZATION: represent the integrated effects
- How do we formally represent this?

Physics -> Parameterizations

- Parameterizations approximate the bulk effects of physical processes too small, too brief, too complex, or too poorly understood to be explicitly represented
- In most modern models, the following parameterizations are used to represent processes to fast or small or even not well known enough….
    - cumulus convection
    - microphysical processes
    - radiation (short wave, long wave)
    - turbulence and diffusive processes
    - boundary layer and surface fluxes
    - interactions with earth’s surface (mountain drag effects)
- Many of the biggest improvements in model forecasts will come from improving these parameterizations

Reynolds Averaging

- Integrating the governing differential equations in a limited area numerically will limit the explicit representation of atmospheric motions and processes at a scale smaller than the grid interval, truncated wavelength, or finite element
- The subgrid-scale disturbances may be inappropriately represented by the grid point values, which may cause nonlinear aliasing and nonlinear numerical instability
- One way to resolve the problem is to explicitly simulate any significant small-scale motions and processes. This is called direct numerical simulation (DNS). This would require grids where Δx ~ 0.1 - 1 m.
- DNS is impractical for NWP. Models now simulate large turbulent eddies explicitly. This is called large-eddy simulations (LES).
- Reynolds averaging is the formalism which separates out the resolvable and unresolvable scales of motion in the equations themselves.
- We do so by splitting our dependent variables (u, T, q, etc.) into mean (resolved) and turbulent (perturbation/unresolved) components, e.g.,

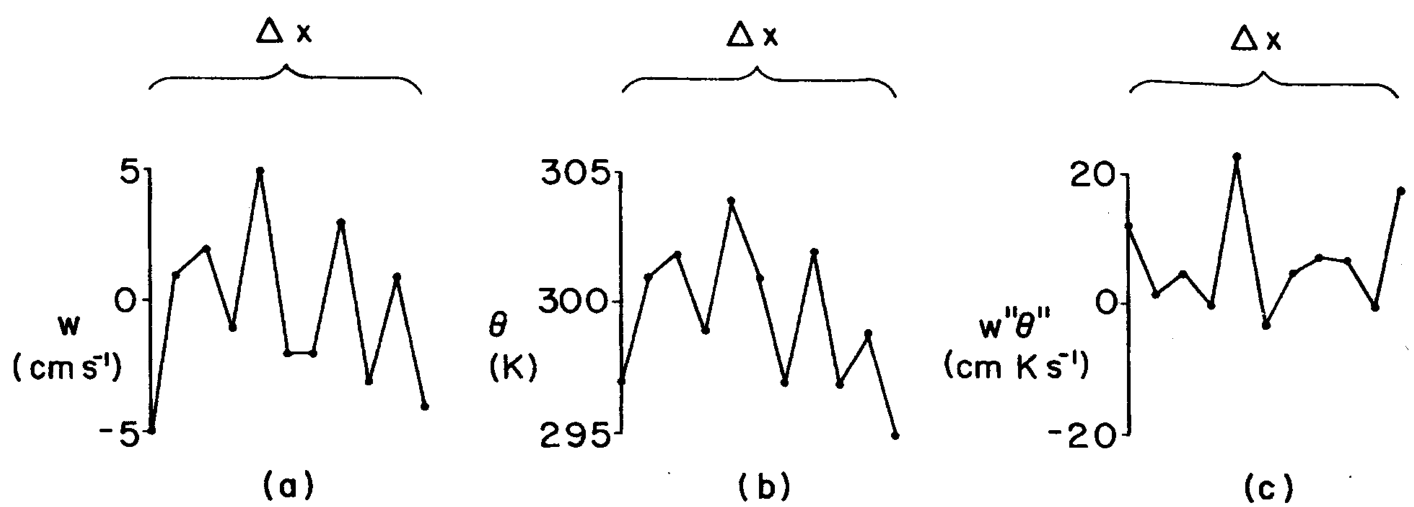
Reynolds Averaging

$$w = \overline{w} + w' \quad \theta = \overline{\theta} + \theta'$$

$$w\theta = \overline{w}\overline{\theta} + \overline{w'\theta'} + \cancel{\overline{\overline{w}\theta'}} + \cancel{\overline{w'\overline{\theta}}}$$

In statistical terms, these fluxes, as an average of the product of deviation components, are also called *covariances*.

Figure shows the subgrid scale covariance $\overline{w'\theta'}$.



Reynolds Averaging for Bnd Layer

$$\frac{D\bar{u}}{Dt} = f\bar{v} - \frac{1}{\rho_o}\frac{\partial \bar{p}}{\partial x} - \frac{1}{\rho_o}\left[\frac{\partial(\rho_o\overline{u'u'})}{\partial x} + \frac{\partial(\rho_o\overline{u'v'})}{\partial y} + \frac{\partial(\rho_o\overline{u'w'})}{\partial z}\right] +$$

$$\frac{D\bar{v}}{Dt} = -f\bar{u} - \frac{1}{\rho_o}\frac{\partial \bar{p}}{\partial y} - \frac{1}{\rho_o}\left[\frac{\partial(\rho_o\overline{u'v'})}{\partial x} + \frac{\partial(\rho_o\overline{v'v'})}{\partial y} + \frac{\partial(\rho_o\overline{v'w'})}{\partial z}\right]$$

$$\frac{D\bar{w}}{Dt} = -\frac{1}{\rho_o}\frac{\partial p_1}{\partial z} - g\frac{\rho_1}{\rho_o} - \frac{1}{\rho_o}\left[\frac{\partial(\rho_o\overline{u'w'})}{\partial x} + \frac{\partial(\rho_o\overline{v'w'})}{\partial y} + \frac{\partial(\rho_o\overline{w'w'})}{\partial z}\right]$$

$$\frac{D\bar{\theta}}{Dt} = \overline{S_\theta} - \frac{1}{\rho_o}\left[\frac{\partial(\rho_o\overline{u'\theta'})}{\partial x} + \frac{\partial(\rho_o\overline{v'\theta'})}{\partial y} + \frac{\partial(\rho_o\overline{w'\theta'})}{\partial z}\right] + \kappa\nabla^2\bar{\theta},$$

$$\frac{D\bar{\phi}}{Dt} = \overline{S_\phi} - \frac{1}{\rho_o}\left[\frac{\partial(\rho_o\overline{u'\phi'})}{\partial x} + \frac{\partial(\rho_o\overline{v'\phi'})}{\partial y} + \frac{\partial(\rho_o\overline{w'\phi'})}{\partial z}\right] + \kappa\nabla^2\bar{\phi},$$

$$\phi = q_v, q_c, q_i, q_r, q_s, q_g,$$

In the above, $\overline{v'\theta'}$, and $\overline{w'\theta'}$ are turbulent heat fluxes, $\overline{u'w'}$ and $\overline{v'w'}$ are vertical turbulent fluxes of zonal momentum, and $\overline{u'v'}$ is the horizontal turbulent flux of zonal momentum.

In order to "close" the system (<u>closure problem</u>), the flux terms need to be represented (<u>parameterized</u>) by the grid-volume averaged terms (terms with "upper bar"s).

Boundary layer approximation
(horizontal scales >> vertical scales), e.g. : $\frac{\partial\overline{u'u'}}{\partial x} \ll \frac{\partial\overline{u'w'}}{\partial z}$

High Reynolds number approximation
(molecular diffusion << turbulent transports), e.g.: $\nu\nabla^2 U \ll \frac{\partial\overline{u'w'}}{\partial z}$

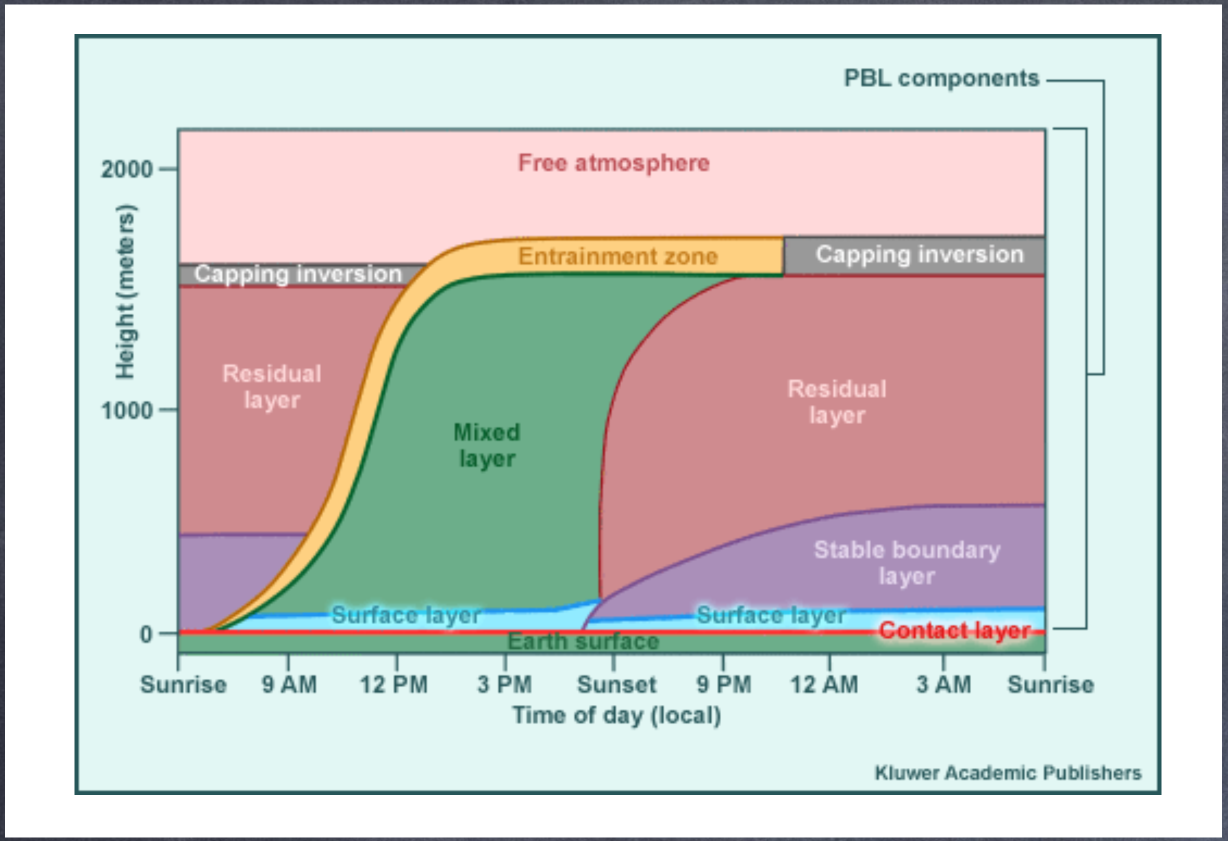
$$\boxed{\begin{aligned}
\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + W\frac{\partial U}{\partial z} - fV &= -\frac{1}{\rho_o}\frac{\partial P}{\partial x} - \frac{\partial\overline{u'w'}}{\partial z} \\
\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + W\frac{\partial V}{\partial z} + fU &= -\frac{1}{\rho_o}\frac{\partial P}{\partial y} - \frac{\partial\overline{v'w'}}{\partial z}
\end{aligned}}$$

Reynolds Stress

Closure Problem

- Estimating those Reynolds stress terms is called the closure problem
  - to close the system of equations to be solved we need to decide how to formulate those fluxes IN TERM OF THE MEAN VARIABLES!
- Various levels of “closure”
  - 1st order (diagnostic closures)
  - 2nd order (prognostic closures)
  - 3rd and higher (here be dragons….)
- <u>*For all closures, you end up with “picking” some coefficients or choosing an approach which approximates some process (often poorly)*</u>

Here comes complexity!



- Planetary Boundary Layer
  - contact layer
  - surface layer
  - boundary layer

- Reynolds fluxes must account for….
  - nocturnal effect
  - stable BL boundary layer
  - neutral BL
  - convective BL
  - capping inversion
  - residual layers
  - ?????

Closure Methods

Bulk Aerodynamic Parameterization

The boundary layer is treated as a single slab and assume the wind speed and potential temperature are independent of height, and the turbulence is horizontally homogeneous.

$$\overline{u'w'} = -C_d \overline{V}^2 \cos\mu\,; \quad \overline{v'w'} = -C_d \overline{V}^2 \sin\mu\,; \quad \overline{w'\theta'} = -C_h \overline{V}^2 \left[\overline{\theta} - \overline{\theta}_{z_o}\right],$$

Cd, Ch now need to be specified!

where C_d and C_h are nondimensional *drag and heat transfer coefficients*, respectively,

K-theory parameterization

In this approach, the turbulent flux terms in (14.1.3)-(14.1.7) are written as,

$$\overline{u'w'} = -K_m \frac{\partial \overline{u}}{\partial z}; \quad \overline{v'w'} = -K_m \frac{\partial \overline{v}}{\partial z}; \quad \overline{w'\theta'} = -K_h \frac{\partial \overline{\theta}}{\partial z}; \quad \overline{w'q'} = -K_q \frac{\partial \overline{q}}{\partial z}. \quad (14.2.1)$$

Km, Kh now need to be specified!

If the gradient terms of (14.2.1) (e.g., $\partial\overline{u}/\partial z$) are calculated based on local gradients, it is call local closure; otherwise it is called non-local closure. Normally, a non-local closure would do a better job for a convective boundary layer.

$$K_m \sim c_m L^2 \left|\frac{\partial \vec{V}}{\partial z}\right|$$

$$K_m \sim c_m L^2 \left(\frac{R_i^c - R_i}{R_i}\right)\left|\frac{\partial V}{\partial z}\right|$$

Turbulent kinetic energy (TKE or 1 1/2) closure scheme

The TKE, $(\overline{u'^2} + \overline{v'^2} + \overline{w'^2})/2$, is predicted, while the other subgrid scale turbulent flux terms are diagnosed and related to the TKE and to the grid-scale mean values.

$$\frac{\partial \overline{e}}{\partial t} = \underset{1}{-\overline{\mathbf{V}} \cdot \nabla \overline{e}} \underset{2}{- \overline{\mathbf{V}' \cdot \nabla e}} \underset{3}{- (1/\rho_o)\left[(\overline{u'p'})_x + (\overline{v'p'})_y + (\overline{w'p'})_z\right]} \underset{4}{- (g/\rho_o)\overline{\rho' w'}}$$

$$\underset{5}{-\left[\left(\overline{u'u'}\,\overline{u}_x + \overline{u'v'}\,\overline{u}_y + \overline{u'w'}\,\overline{u}_z\right) + \left(\overline{u'v'}\,\overline{v}_x + \overline{v'v'}\,\overline{v}_y + \overline{v'w'}\,\overline{v}_z\right) + \left(\overline{u'w'}\,\overline{w}_x + \overline{v'w'}\,\overline{w}_y + \overline{w'w'}\,\overline{w}_y\right)\right]} \underset{6}{+ \nu\nabla^2 \overline{e}} \underset{7}{- \nu\left(\overline{u'^2_x} + \overline{v'^2_y} + \overline{w'^2_z}\right)} \quad (14.2.31)$$

$$K_m \sim c_m L \sqrt{\overline{e}}$$

TKE Closure

local TKE: $E' \equiv 1/2(u'^2 + v'^2 + w'^2)$

mean TKE: $E \equiv 1/2(\overline{u'^2} + \overline{v'^2} + \overline{w'^2})$

Derive equation for E by combining equations of total velocity components and mean velocity components:

Storage

Mean flow TKE advection

$$\frac{\partial E}{\partial t} + U\frac{\partial E}{\partial x} + V\frac{\partial E}{\partial y} + W\frac{\partial E}{\partial z} =$$

Pressure correlation

$$-\frac{\partial}{\partial z}\overline{E'w'} - \overline{u'w'}\frac{\partial U}{\partial z} - \overline{v'w'}\frac{\partial V}{\partial z} - \frac{g}{\rho_o}\overline{\rho'w'} + \frac{\partial}{\partial z}\frac{\overline{p'w'}}{\rho} - \varepsilon$$

Turbulent transport

Shear production

Buoyancy

Dissipation

You still have to close buoyancy (include effects of moisture), pressure and TKE dissipation terms!

Parameterization of Moist Processes

In most mesoscale and NWP models, the majority of clouds, especially convective clouds, cannot be resolved by grid mesh and the moist variables need to be parameterized by the grid-volume mean variables.

Although in cloud models, the resolution is fine enough to roughly represent the clouds, the microphysical processes still need to be parameterized or properly represented.

The treatments of moist processes in a mesoscale model into two categories: (1) *parameterization of microphysical processes*, and (2) *cumulus parameterization*.

For parameterization of microphysical processes, two approaches have been taken: (a) explicit representation, and (b) bulk parameterization (normally referred to grid explicit microphysics, which is different from (a)).

Cumulus Parameterization

The collective effects of cumulus clouds at subgrid scale, such as the convective condensation and transport of heat, moisture, and momentum, on the larger scale environment are essential and need to be represented by grid-scale variables.

On the other hand, the large-scale forcing tends to modulate the cumulus convection, which in turn determines the total rainfall rate.

The representation of these processes is carried out by the *cumulus parameterization schemes*.

To parameterize the interaction between cumulus clouds and their environment, we must determine the relationship between cumulus convection and its larger-scale environment.

Cumulus parameterization schemes may be divided into schemes for large-scale models ($\Delta x > 50km; \Delta t > O(\text{min})$) and schemes for mesoscale models ($10km < \Delta x < 50km; \Delta t < O(\text{min})$).

For models having grid spacing less than 10 km, microphysics parameterization schemes are more appropriate and often employed.

Explicit Microphysics

In the bulk parameterization approach, each category of the water substance is governed by its own continuity equation.

The shape and size distributions are assumed a priori and the basic microphysical processes are parameterized.

The water substance may be divided into six categories: (1) water vapor, (2) cloud water, (3) cloud ice, (4) rain, (5) snow, and (6) graupel/hail (Orville 1980; Lin, Farley, and Orville 1983 - LFO scheme or Lin et al. scheme).

Some basic microphysical processes:

Accretion: Any larger precipitation particle overtakes and captures a smaller one.

Coalescence: The capture of small cloud droplets by larger cloud droplets or raindrops.

Autoconversion: The initial stage of the collision–coalescence process whereby cloud droplets collide and coalesce to form drizzle drops.

Aggregation: The clumping together of ice crystals to form snowflakes.

Riming: Droplets freeze immediately on contact of ice crystal will form **rimed crystal** or **graupel**. If freezing is not immediate, it may form **hail**.

The *size distributions* of rain (q_r), snow (q_s), and graupel or hail (q_g) are hypothesized as

$$N_k(D) = N_{ok}\exp(-\lambda_k D_k)\,, \qquad (14.3.6)$$

where $k = r, s, or\ g$, N_{ok} is based on observations, D_k is the diameter of the water substance, and λ_p is the *slope parameter* of the size distribution.

This type of distribution is called the *Marshall-Palmer distribution* (Marshall and Palmer 1948).

The slope parameters are given by

$$\lambda_k = \left(\frac{\pi\rho_k N_{ok}}{\rho q_k}\right)^{0.25},$$

where ρ_k is the density of water, snow or graupel.

In general, the size distribution (14.3.6) includes the shape factor and is written as

$$N_k(D) = N_{ok}D_k^{\alpha}\exp(-\lambda_k D_k)\,,\ k = r,\ s,\ or\ g, \qquad (14.3.10)$$

where α is called the shape parameter. Thus, there are 3 parameters or moments, N_{ok}, λ_k, α, to be determined.

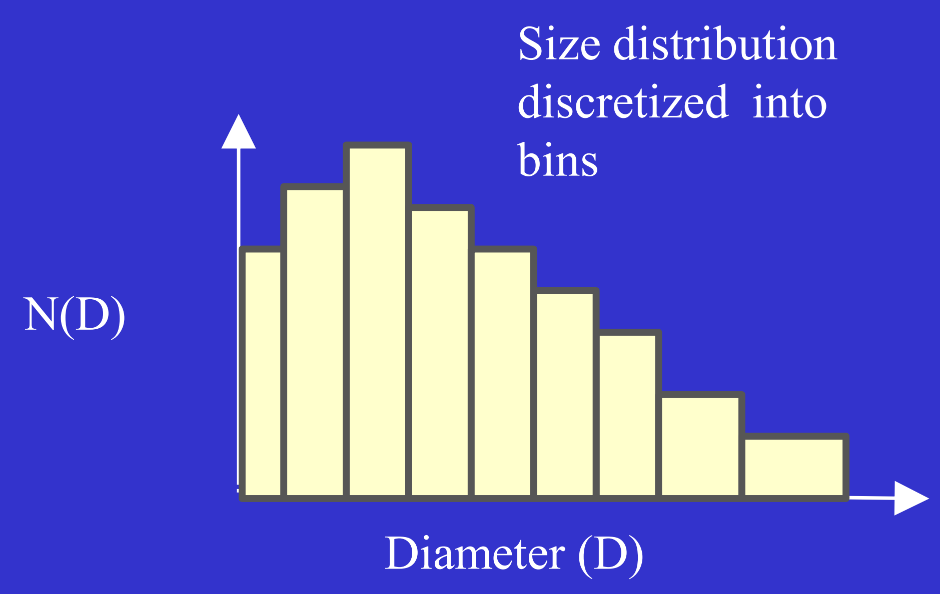
Microphysical Schemes

- Various levels of complexity
- Single moment
  - predict mixing ratio (lambda)
  - Fix N_0, alpha (impacts reflectivity factor Z)
- Double moment
  - predict mixing ratio, N_0
  - alpha is fixed
- “2.5” scheme: diagnose alpha from mean variables and type of particle
- 3 moment - predict q, N_0 and Z.
- Bin models
  - break distribution into “bins” (like 100-200 bins)
  - prediction of interactions between all bins
  - just now feasible for water and ice in 3D cloud models (Ted Manselll)

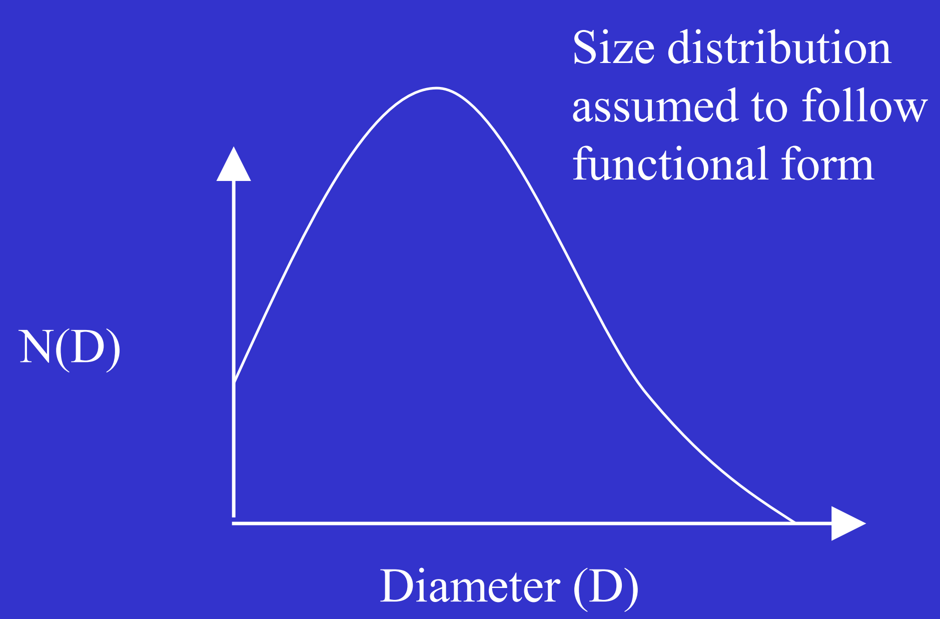
Examples

Microphysics schemes can be broadly categorized into two types:

Detailed (bin)



bulk



Representation of particle size distribution

1 Mom. Microphysical Parameterizations

The microphysical processes are very complicated, which are summarized in Fig. 14.6. (From Lin et al. 1983 – the Lin-Farley-Orville Scheme; MM5 Goddard scheme and several other schemes are based on LFO scheme)

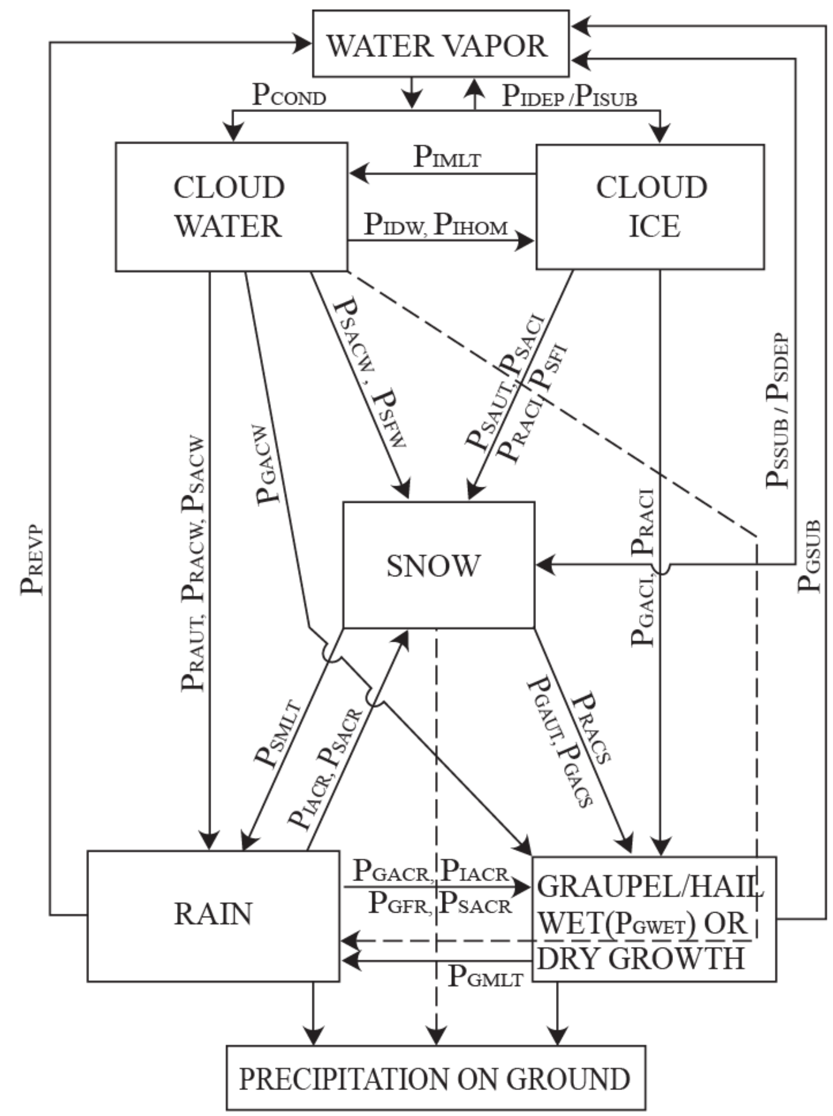


Fig. 14.6: A sketch of cloud microphysical processes in a bulk microphysics parameterization (LFO) scheme including ice phase. Meanings of the production terms (i.e., P terms) can be found in Table 14.1. (Adapted after Lin, Farley, and Orville 1983; Orville and Kopp 1977) (Lin 2007)

2 Mom. Microphysical Parameterizations

Continuity Equations

The prognostic equations for the mixing ratios of all phases of water in the parameterization (i.e., vapor, liquid, ice, and liquid water on ice) are as follows:

$$\frac{dq_v}{dt} = -\mathrm{QCND} - \mathrm{QREVP} - (1-\delta)(\mathrm{QSEVP} + \mathrm{QGEVP} + \mathrm{QHEVP}) - \delta(\mathrm{QINT} + \mathrm{QIDEP} + \mathrm{QSDEP} + \mathrm{QGDEP} + \mathrm{QHDEP}), \quad \text{(A.1)}$$

$$\frac{dq_w}{dt} = \mathrm{QCND} - \mathrm{QRAUT} - \mathrm{QRACW} - \mathrm{QSACWS} - \mathrm{QGACWG} - \mathrm{QHACWH} - \mathrm{QIFM} - \delta(\mathrm{QIACW} + \mathrm{QIHR} + \mathrm{QSACWG} + \mathrm{QGACWH} + \mathrm{QHACWG}), \quad \text{(A.2)}$$

$$\frac{dq_i}{dt} = \mathrm{QIFM} + \delta(\mathrm{QINT} + \mathrm{QIDEP} + \mathrm{QIACW} + \mathrm{QIHR} + \mathrm{QIHMS} + \mathrm{QIHMG} + \mathrm{QIHMH} - \mathrm{QICNVS} - \mathrm{QRACI} - \mathrm{QSACI} - \mathrm{QGACI} - \mathrm{QHACI}), \quad \text{(A.3)}$$

$$\frac{dq_r}{dt} = \mathrm{QREVP} + \mathrm{QRAUT} + \mathrm{QRACW} + \mathrm{QSSHD} + \mathrm{QGSHD} + \mathrm{QHSHD} - \delta(\mathrm{QIACR} + \mathrm{QSACRS} + \mathrm{QSACRG} + \mathrm{QSACRH} + \mathrm{QGACRG} + \mathrm{QGACRH} + \mathrm{QHACR}), \quad \text{(A.4)}$$

$$\frac{dq_s}{dt} = \mathrm{QSACWS} - \mathrm{QGACS} - \mathrm{QHACS} - \mathrm{QSSHD} + (1-\delta)\mathrm{QSEVP} + \delta(\mathrm{QSDEP} + \mathrm{QICNVS} + \mathrm{QSACI} + \mathrm{QSACRS} - \mathrm{QRACSG} - \mathrm{QRACSH} - \mathrm{QWACSG} - \mathrm{QIHMS}), \quad \text{(A.5)}$$

$$\frac{dq_g}{dt} = \mathrm{QGACWG} + \mathrm{QGACS} - \mathrm{QGSHD} + (1-\delta)\mathrm{QGEVP} + \delta(\mathrm{QGDEP} + \mathrm{QGACI} + \mathrm{QGACRG} + \mathrm{QSACRG} + \mathrm{QRACSG} + \mathrm{QSACWG} + \mathrm{QWACSG} + \mathrm{QHACWG} + \mathrm{QWACHG} - \mathrm{QRACGH} - \mathrm{QWACGH} - \mathrm{QIHMG}), \quad \text{(A.6)}$$

$$\frac{dq_h}{dt} = \mathrm{QHACWH} + \mathrm{QHACS} - \mathrm{QHSHD} + (1-\delta)\mathrm{QHEVP} + \delta(\mathrm{QHDEP} + \mathrm{QHACI} + \mathrm{QHACR} + \mathrm{QIACR} + \mathrm{QRACI} + \mathrm{QSACRH} + \mathrm{QRACSH} + \mathrm{QGACRH} + \mathrm{QRACGH} + \mathrm{QGACWH} + \mathrm{QWACGH} - \mathrm{QWACHG} - \mathrm{QIHMH}), \quad \text{(A.7)}$$

$$\frac{dq_{sw}}{dt} = \mathrm{QSACW} - \mathrm{QSFM} - \mathrm{QSSHD} - F_{sw}(\mathrm{QGACS} + \mathrm{QHACS}) + (1-\delta)\mathrm{QSEVP} + \delta[\mathrm{QSACRS} - F_{sw}(\mathrm{QRACSG} + \mathrm{QRACSH} + \mathrm{QWACSG})], \quad \text{(A.8)}$$

$$\frac{dq_{gw}}{dt} = \mathrm{QGACW} - \mathrm{QGFM} - \mathrm{QGSHD} + F_{sw}\cdot\mathrm{QGACS} + (1-\delta)\mathrm{QGEVP} + \delta[\mathrm{QGACRG} + \mathrm{QSACRG} + \mathrm{QSACWG} + \mathrm{QHACWG} + F_{sw}(\mathrm{QRACSG} + \mathrm{QWACSG}) + F_{hw}\cdot\mathrm{QWACHG} - F_{gw}(\mathrm{QRACGH} + \mathrm{QWACGH})], \quad \text{(A.9)}$$

$$\frac{dq_{hw}}{dt} = \mathrm{QHACW} - \mathrm{QHFM} - \mathrm{QHSHD} + F_{sw}\cdot\mathrm{QHACS} + (1-\delta)\mathrm{QHEVP} + \delta[\mathrm{QIACR} + \mathrm{QSACRH} + \mathrm{QGACRH} + \mathrm{QHACR} + \mathrm{QGACWH} + F_{sw}\cdot\mathrm{QRACSH} + F_{gw}(\mathrm{QRACGH} + \mathrm{QWACGH}) - F_{hw}\cdot\mathrm{QWACHG}]. \quad \text{(A.10)}$$

The functions δ in (A.1)–(A.10) and F_{xw} in (A.8)–(A.10) are defined as

$$\delta = \begin{cases} 1, & T < 0^\circ\mathrm{C} \\ 0, & \text{otherwise,} \end{cases} \quad \text{(A.11)}$$

$$F_{xw} = q_{xw}/q_x, \quad \text{(A.12)}$$

where the variable x represents the precipitation ice species of snow, graupel, and hail/frozen drops ($x = s, g, h$).

Changes in the simulated potential temperature (q) due to latent heating are calculated using the following thermodynamic energy equation:

$$\frac{d\theta}{dt} = \frac{L_v}{\Pi C_p}(\mathrm{QCND} + \mathrm{QREVP}) + \frac{\delta L_s}{\Pi C_p}(\mathrm{QINT} + \mathrm{QIDEP} + \mathrm{QSDEP} + \mathrm{QGDEP} + \mathrm{QHDEP}) + \frac{L_f}{\Pi C_p}[\mathrm{QIFM} + \mathrm{QSFM} + \mathrm{QGFM} + \mathrm{QHFM} + \delta(\mathrm{QIACW} + \mathrm{QIHR})], \quad \text{(A.13)}$$

where Π is the Exner function $(p_0/p)^\kappa$ and $\kappa = R_d/C_p$.

Finally, prognostic equations for the number concentrations of each ice species are

$$\frac{dn_i}{dt} = \mathrm{NIFM} + \delta(\mathrm{NINT} + \mathrm{NIDEP} + \mathrm{NIHMS} + \mathrm{NIHMG} + \mathrm{NIHMH} + \mathrm{NIHR} - \mathrm{NICNV} - \mathrm{NIACI} - \mathrm{NRACI} - \mathrm{NSACI} - \mathrm{NGACI} - \mathrm{NHACI}), \quad \text{(A.14)}$$

$$\frac{dn_s}{dt} = \mathrm{NSBR} - \mathrm{NSACS} - \mathrm{NGACS} - \mathrm{NHACS} + (1-\delta)(\mathrm{NSEVP} - \mathrm{NSSHD}) + \delta(\mathrm{NSCNV} + \mathrm{NSDEP} - \mathrm{NRACSG} - \mathrm{NRACSH} - \mathrm{NWACSG}), \quad \text{(A.15)}$$

$$\frac{dn_g}{dt} = (1-\delta)(\mathrm{NGEVP} - \mathrm{NGSHD}) + \delta(\mathrm{NGDEP} + \mathrm{NSACRG} + \mathrm{NWACSG} + \mathrm{NWACHG} - \mathrm{NRACGH} - \mathrm{NWACGH}), \quad \text{(A.16)}$$

$$\frac{dn_h}{dt} = (1-\delta)(\mathrm{NHEVP} - \mathrm{NHSHD}) + \delta(\mathrm{NHDEP} + \mathrm{NIACR} + \mathrm{NSACRH} + \mathrm{NGACRH} + \mathrm{NWACGH} - \mathrm{NWACH}). \quad \text{(A.17)}$$

Ferrier JAS 1994

NWP in a week…summary…

- Basic equations
  - 3 forms of compressible equations
  - various approximations to equations (fully compressible, hydrostatic, anelastic)
- Horizontal grids
  - global, local
  - grid point, finite volume, structured FV, unstructured FV, spectral
- Vertical grids
  - types (z, p, sigma)
  - coordinate transforms
  - errors associated with coordinate transform: e.g., PGF
- Choices driving model choices: (problem to be solved, efficiency versus accuracy, etc)
- Numerical Methods
  - CFL criteria
  - Taylor series analysis
  - stability analysis
- Parameterizations
  - radiation
  - microphysics
  - land surface

The end….

for now….

Thanks for the opportunity to teach!