The Value of Climatological, Categorical and Probabilistic Forecasts in the Cost-Loss Ratio Situation

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ABSTRACT

This paper is concerned with the value of climatological, categorical, probabilistic and perfect forecasts in the cost-loss ratio situation. Expressions are derived for the expense associated with these different types of forecasts, and measures of value and relative value are formulated in terms of these expressions. Some relationships among these expressions and measures are described, and these relationships are illustrated by examining both hypothetical and real sets of forecasts.

It is demonstrated that, if the probabilistic forecasts of concern are (completely) reliable, then the value of these forecasts is greater than the value of climatological and categorical forecasts for all activities or operations (i.e., for all values of the cost-loss ratio C/L). On the other hand, if the forecasts are unreliable, then the value of climatological and/or categorical forecasts may be greater than the value of probabilistic forecasts for some values of C/L. However, examination of hypothetical and real sets of unreliable forecasts indicates that the relationships between the value of *reliable* probabilistic forecasts and the value of climatological and categorical forecasts are quite robust in the sense that these relationships appear to hold for most if not all values of C/L even for moderately unreliable forecasts.

The results presented in this paper have important implications for operational forecasting procedures and practices. These implications relate to the desirability of formulating and disseminating a wide variety of weather forecasts in probabilistic terms and of achieving and maintaining a high degree of reliability in probabilistic forecasts.

1. Introduction

The value of weather forecasts is a topic of considerable importance, and applied meteorologists, economists and others have attempted to assess the value of such forecasts in a variety of contexts (e.g., Thompson and Brier, 1955; Kolb and Rapp, 1962; Lave, 1963; Nelson and Winter, 1964; Thompson, 1972; Anderson, 1973; Howe and Cochrane, 1976). In a particular context or decision-making situation, the value of weather forecasts depends upon many factors, including the nature of the forecasts themselves. With regard to the latter, several different types of forecasts can be identified: 1) climatological forecasts (e.g., forecasts based upon climatological probabilities); 2) categorical or deterministic forecasts; and 3) probabilistic forecasts.[2] Since these three types of forecasts are all currently available and could be disseminated to potential users and decision makers, it would seem to be particularly important to determine *and* compare the value of these different types of forecasts, both in general and in specific decision-making situations.

The so-called "cost-loss ratio situation" is a decision-making situation frequently encountered in the meteorological literature. This situation involves a decision maker who must decide whether or not to take protective action in the face of uncertainty as to whether or not adverse weather will occur. In the original model of this situation (Thompson, 1952; see also Section 2 of this paper), the decision maker incurs a cost (of protection) if he takes protective action and he suffers a loss if he does not take protective action and adverse weather subsequently occurs. While this model is a very simple normative model, it appears to provide a realistic description of situations faced by many forecast-sensitive decision makers, and, as a result, the model has been used extensively by meteorologists and others in both real and hypothetical decision-making situations (e.g., Thompson, 1952; Kolb and Rapp, 1962; Allen and Lambert, 1971a, b; Kernan, 1975). Moreover, more complex decision-making situations can, in some cases, be investigated using this model, either by simplifying the situations or by general-

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[2] These three types of forecasts do not, of course, form a mutually exclusive or collectively exhaustive set. In this regard, forecasts based upon climatological probabilities are also probabilistic forecasts and forecasts based upon persistence or some combination of persistence and climatology are not included in this set. However, they do represent three of the principal types of weather forecasts which are currently produced on a routine, operational basis.

izing the model (e.g., Nelson and Winter, 1964; Epstein, 1969; Howe and Cochrane, 1976; Murphy, 1976).

Some 25 years ago, Thompson (1952) showed that the expense (i.e., costs and/or losses to decision makers) associated with the use of quite accurate categorical forecasts in the cost-loss ratio situation could exceed the expense associated with forecasts based solely upon sample climatological probabilities. This result demonstrated, for the first time, the potential value of expressing weather information in probabilistic terms.[3] More recently, Thompson (1962, 1972) has investigated the (potential) economic benefits associated with "operational improvements" and "scientific advances" in weather forecasting within the context of the cost-loss ratio situation. In this terminology, gains related to operational improvements refer to the benefits associated with the formulation and use of probabilistic forecasts (based on current levels of forecasting ability), while gains related to scientific advances refer to the additional benefits which could be realized if the forecasts which resulted from such advances always led to "optimal" decisions. While Thompson's recent studies are closely related to the work to be described in this paper, these studies were primarily concerned with estimating the economic gains associated with these improvements and advances, rather than with comparing the gains or expenses associated with different types of forecasts. In this study we investigate—from a theoretical point of view—the relationships among the expenses associated with the use (by decision makers) of different types of forecasts, including the conditions under which such relationships hold, and then consider the practical implications of these results by examining hypothetical and real sets of forecasts.

The purposes of this paper are to describe some relationships among the expenses associated with the use of climatological, categorical, probabilistic and perfect forecasts within the framework of the cost-loss ratio situation and to illustrate these relationships by examining both hypothetical and real sets of forecasts. The cost-loss ratio situation itself is briefly described in Section 2. Some basic expressions for and relationships among the expenses associated with these different types of forecasts are derived in Section 3, and these relationships are illustrated in Section 4 using hypothetical and real sets of forecasts. Section 5 consists of a summary and conclusion, including a discussion of some implications of the results of this study for operational forecasting procedures and practices.

[3] That paper, however, did not consider the expense associated with day-to-day probabilistic forecasts. Its purpose was not to compare the value of categorical and probabilistic forecasts, but rather to determine the accuracy—defined in terms of the percentage of correct categorical forecasts—that would be required for the expense associated with such forecasts to be equal to the expense associated with forecasts based solely upon sample climatological probabilities.

2. The cost-loss ratio situation

The cost-loss ratio situation involves a decision maker who must decide whether or not to take protective action, with respect to some activity or operation, in the face of uncertainty as to whether or not weather adverse to the activity will occur. Specifically, the decision maker has two possible actions, "protect" and "do not protect," and two weather events can occur, "adverse weather" and "no adverse weather." In the terminology of decision analysis, this situation is a two-action, two-state decision-making situation. We shall denote the actions by a_1 (protect) and a_2 (do not protect) and the events or states by s_1 (adverse weather) and s_2 (no adverse weather). Each action-state pair $\{a_m, s_n\}(m, n=1, 2)$ leads to a different outcome or consequence. If we denote the consequences by $o_{mn}(m, n=1, 2)$, then $o_{11}=\{$protect, adverse weather$\}$, $o_{12}=\{$protect, no adverse weather$\}$, $o_{21}=\{$do not protect, adverse weather$\}$, and $o_{22}=\{$do not protect, no adverse weather$\}$.

The payoffs associated with the consequences in the cost-loss ratio situation are generally expressed in terms of the monetary expenses of the decision maker. In the original model of this situation (Thompson, 1952), the cost of protection is denoted by C and it is assumed that, when protective action is taken, the activity is completely protected against the effects of adverse weather. Thus, if we denote the expense associated with consequence o_{mn} by $e_{mn}(m, n=1, 2)$, then $e_{11}=e_{12}=C$. Further, if the loss which results when protective action is not taken and adverse weather occurs is denoted by L, then $e_{21}=L$. Finally, since no loss results in this latter case when adverse weather does not occur, $e_{22}=0$ (zero). The expense matrix $\mathbf{E}=(e_{mn})$ $(m, n=1, 2)$ associated with this model of the cost-loss ratio situation is depicted in Table 1. It can be assumed, without any loss of generality, that the bounds on the values of C and L are such that $0 \leqslant C \leqslant L < \infty$.

Within the framework of the cost-loss ratio situation, the weather forecast of concern is a probability vector $\mathbf{r}=(r_1, r_2)$ $(r_n \geqslant 0,\ r_1+r_2=1)$, in which r_n is the probability of occurrence of state $s_n (n=1, 2)$. Since $r_2=1-r_1$, we can use the symbol r without a subscript to denote r_1 (i.e., $r_1=r$, $r_2=1-r$). It will be assumed that the

TABLE 1. The cost-loss ratio situation, including the expense matrix $\mathbf{E}$ and the expected expense E_m $(m=1, 2)$.

		Expense matrix $\mathbf{E}$ States		
		s_1 (Adverse weather)	s_2 (No adverse weather)	Expected expense (E)
Actions	a_1 (Protect)	C	C	$E_1=r_1C+r_2C=C$
	a_2 (Do not protect)	L	0	$E_2=r_1L$
Probability vector	$\mathbf{r}$	r_1	r_2	

decision maker uses the forecasts in their original form (i.e., that he does not modify the forecasts). The forecasts may be produced by objective or subjective forecasting procedures, and, as a result, the probability (r) can be interpreted as a relative frequency or as a subjective degree of belief (see, e.g., Murphy and Winkler, 1971).

We shall assume that the decision maker wants to select the action which minimizes his expected expense.[4] If the expected expense associated with action a_m in this model is denoted by $E_m (m=1, 2)$, then $E_1 = rC+(1-r)C=C$ and $E_2=rL$ (see Table 1). Thus, the decision maker should select action $a_1(a_2)$ when the probability r of adverse weather is greater (less) than the cost-loss ratio C/L, and he is indifferent between the two actions (i.e., $E_1=E_2$) when $r=C/L$.[5] Thus, the decision rule in this situation can be summarized as follows: "protect" if $r \geqslant C/L$ and "do not protect" if $r<C/L$. This rule describes the way in which the forecast (r) should be used by the decision maker to minimize his expected expense within this framework.

3. Value of climatological, categorical and probabilistic forecasts: Some expressions and relationships

a. Some preliminaries

1) The forecasts

We shall use the symbol p to denote an individual probabilistic forecast, where p is the probability of adverse weather (the probability of no adverse weather is then $1-p$). In addition, we shall assume that p can assume only a finite number of K distinct values $p_k (k=1, \ldots, K)$. Then a set of n probabilistic forecasts can be divided into K subsets, where the kth subset consists of the n_k forecasts for which

$$p=p_k(\sum_k n_k=n; k=1, \ldots, K).$$

The categorical forecasts of concern in this paper are derived from the probabilistic forecasts by comparing the forecast probabilities $p_k (k=1, \ldots, K)$ with some critical probability value p^*. Specifically, $p_k \geqslant p^*$ would lead to categorical forecasts of adverse weather, while $p_k<p^*$ would lead to categorical forecasts of no adverse weather. A value of p^* which is frequently used to transform probabilistic forecasts into categorical forecasts in two-state situations is 0.5.

The climatological forecasts of concern here are based upon *sample* climatological probabilities and, as a result, they can be derived from the set of observations which corresponds to the basic set of (probabilistic) forecasts.[6] In this regard, we use the symbol d to denote an individual observation, where $d=1$ if adverse weather occurs and $d=0$ otherwise. Further, we shall denote the relative frequency of adverse weather for the subset of forecasts for which $p=p_k$ by $\bar{d}_k$. Then, the sample climatological probability of adverse weather is p^c, where

$$p^c=(1/n)\sum_k n_k\bar{d}_k \quad (k=1, \ldots, K).$$

Finally, a perfect forecast is a probabilistic forecast for which the probability p is one if adverse weather occurs and zero otherwise (i.e., for which $p=d$). A set of n perfect forecasts would consist, in general, of two subsets, a subset of n_1 forecasts for which $p_1=\bar{d}_1=0$ and a subset of n_2 forecasts for which $p_2=\bar{d}_2 =1 (n_1+n_2=n)$.

2) Reliable and unreliable probabilistic forecasts

In this study we will be interested in whether or not the sets of probabilistic forecasts of concern are reliable or unreliable. In this regard, a set of n probabilistic forecasts will be said to be completely reliable if and only if the forecast probability corresponds exactly to the observed relative frequency for each subset of forecasts; i.e., if and only if $p_k=\bar{d}_k$ for all k ($k=1, \ldots, K$). On the other hand, the set of forecasts will be said to be unreliable if the forecast probability does not correspond to the observed relative frequency for one or more subsets of forecasts; i.e., if $p_k \neq \bar{d}_k$ for some k ($k=1, \ldots, K$). Empirical studies involving the evaluation of probabilistic forecasts for a variety of weather variables have shown that such forecasts are, in general, quite reliable (see, e.g., Root, 1962; Hughes, 1965; Roberts *et al.*, 1969; Sanders, 1973; Sadowski and Cobb, 1974; Bosart, 1975; Murphy and Winkler, 1977). However, a set of real probabilistic forecasts will seldom if every be (completely) reliable.[7]

b. Basic expense expressions

In this subsection we formulate expressions for the expenses (to decision makers) associated with the use

[4] This assumption is equivalent to assuming that the decision maker's utilities (i.e., his preferences for the consequences) are linearly related to the expenses.

[5] Since $0 \leqslant C \leqslant L < \infty$, $0 \leqslant C/L \leqslant 1$. However, if $C/L=0$, then $C=0 (L>0)$ and the decision maker would always protect. On the other hand, if $C/L=1$, then $C=L$ and the decision maker would never protect. Thus, the "effective range" of the cost-loss ratio C/L is the open unit interval (0, 1).

[6] In effect, we are assuming that the long-term and sample climatological probabilities are equal. If they are not equal, then the expense associated with climatological forecasts (which must necessarily be based upon long-term climatological probabilities) would actually be greater than that given by the expression for E(CLIM) for some values of C/L (see Section 3b). Moreover, since E(CLIM) serves as the zero point on the value scale (see Section 3c), the value of categorical, probabilistic and perfect forecasts is underestimated, in general, in this paper.

[7] Hereafter the term "reliable" will be taken to mean completely reliable.

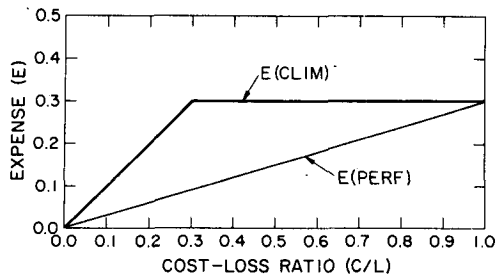

FIG. 1. The basic expense diagram in the cost-loss ratio situation, when the sample climatological probability p^c is equal to 0.3.

of climatological, categorical, probabilistic and perfect forecasts. Hereafter, the term expense will be defined as the *expense per unit loss per forecast.* To determine this standardized expense, the total expense for the set of forecasts of concern will be divided by the loss (L) and the number of forecasts (n). Thus, the expected expenses associated with taking and not taking protective action on n occasions, in standardized terms, are C/L and r, respectively (cf. Table 1).

1) Climatological forecasts

When a climatological forecast p^c is available to the decision maker, he will always protect if $p^c \geqslant C/L$ and he will never protect if $p^c < C/L$. Thus, the (standardized) expense associated with climatological forecasts is E(CLIM), where

$$E(\text{CLIM}) = \begin{cases} C/L & \text{if} \quad p^c \geqslant C/L \\ p^c & \text{if} \quad p^c < C/L. \end{cases} \tag{1}$$

E(CLIM) is a nondecreasing, piecewise linear function of C/L, and this function is depicted in an "expense diagram" in Fig. 1 for $p^c = 0.3$.[8] This diagram, in which expense is plotted as a function of C/L, provides a basic framework within which many of the results described in this paper will be presented.

2) Perfect forecasts

If perfect forecasts were available, then the decision maker would protect when adverse weather was forecast and he would not protect when no adverse weather was forecast. For a set of such forecasts, the decision maker's *total* expense would be the cost of protection (C) multiplied by the number of occasions on which adverse weather was forecast (and observed). Thus, the (standardized) expense associated with perfect forecasts E(PERF) is

$$E(\text{PERF}) = p^c(C/L) \tag{2}$$

(for perfect forecasts, p^c is the "rate" at which adverse weather is forecast *and* observed).[9] E(PERF) is an increasing linear function of C/L, and this function is also depicted in Fig. 1 for $p^c = 0.3$. It should be noted that E(PERF) represents a lower bound on the expense associated with *all* types of forecasts.

3) Categorical forecasts

The *total* expense associated with a set of categorical forecasts is simply the cost of protection (C) multiplied by the number of occasions on which protective action is taken (i.e., the number of occasions on which $p_k \geqslant p^*$) plus the loss (L) multiplied by the number of occasions on which protective action is not taken (i.e., the number of occasions on which $p_k < p^*$) *and* adverse weather subsequently occurs. Thus, the (standardized) expense associated with a set of n categorical forecasts, E(CAT), is

$$E(\text{CAT}) = (1/n)\Big[(C/L) \sum_{\substack{k \\ p_k \geqslant p^*}} n_k + \sum_{\substack{k \\ p_k < p^*}} n_k \bar{d}_k\Big]. \tag{3}$$

E(CAT) is an increasing linear function of C/L, and, for a particular set of forecasts, it can be represented by a line in an expense diagram such as Fig. 1 (see Section 4).

4) Probabilistic forecasts

An expression for the (standardized) expense associated with a set of n probabilistic forecasts, E(PROB), can be "derived" in a manner similar to that used to obtain an expression for E(CAT), except that the decision as to whether or not to take protective action (and thus to incur costs and possibly suffer losses) depends upon C/L rather than the critical probability value p^*. Thus,

$$E(\text{PROB}) = (1/n)\Big[(C/L) \sum_{\substack{k \\ p_k \geqslant C/L}} n_k + \sum_{\substack{k \\ p_k < C/L}} n_k \bar{d}_k\Big]. \tag{4}$$ [10]

Since C/L appears in the limits of the summations, E(PROB) is a *nonlinear* function of C/L, and, for a particular set of forecasts, it can be represented by a curve in an expense diagram such as Fig. 1 (see Section 4).[11]

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[8] Specifically, E(CLIM) is an increasing linear function of C/L for $C/L \leqslant p^c$ and a constant function for $C/L > p^c$.

[9] Of course, for a given value of C/L, perfect forecasts are not required to "reach" this lower bound on expense. In particular, the expense associated with a set of probabilistic forecasts for which $p \geqslant C/L$ when adverse weather occurs and $p < C/L$ when no adverse weather occurs, will be equal to E(PERF). However, only perfect forecasts can attain this minimum expense for *all* values of C/L.

[10] $E(\text{PROB}) = p^c$ when $C/L = 1$.

[11] Since the set of probability values $p_k (k = 1, \ldots, K)$ is discrete, E(PROB) is actually a piecewise linear function of C/L. Specificially, E(PROB) is a linear function of C/L between p_k and $p_{k+1} (k = 1, \ldots, K-1)$, and the slope of these K linear segments decreases, in general, as C/L increases. As the number of probability values (or segments) K increases, E(PROB) "approaches" a nonlinear function of C/L.

c. Some value expressions

In this paper the value of a set of perfect (categorical, probabilistic) forecasts will be defined as the expense associated with climatological forecasts minus the expense associated with perfect (categorical, probabilistic) forecasts. Thus, the expense associated with climatological forecasts represents the zero point on the value scale.

1) PERFECT FORECASTS

The value of perfect forecasts, $V(\text{PERF})$, is simply the difference between $E(\text{CLIM})$ and $E(\text{PERF})$, or, from (1) and (2),

$$V(\text{PERF})=\begin{cases}(C/L)(1-p^c) & \text{if} \quad p^c \geqslant C/L \\ p^c[1-(C/L)] & \text{if} \quad p^c < C/L.\end{cases} \tag{5}$$

$V(\text{PERF})$, a linear function of C/L, is greater than or equal to zero for all values of C/L, with equality only if $C/L=0$ or 1 (assuming that, in general, $0<p^c<1$). Further, $V(\text{PERF})$ is a maximum when $C/L=p^c$. Thus, $V(\text{PERF})_{\text{MAX}}=p^c(1-p^c)$, and this maximum value cannot exceed 0.25 (when $C/L=p^c=0.5$), or one-fourth of the loss L. $V(\text{PERF})$ is depicted as a function of C/L in a "value diagram" in Fig. 2 for $p^c=0.3$, and this function represents an upper bound on the value of *all* other types of forecasts, including categorical and probabilistic forecasts. Thus, the functions defining the value of categorical and probabilistic forecasts cannot exceed the piecewise linear function defining $V(\text{PERF})$ in Fig. 2 (when $p^c=0.3$).

2) CATEGORICAL FORECASTS

The value of categorical forecasts, $V(\text{CAT})$, is, from (1) and (3),

$$V(\text{CAT})=\begin{cases}(1/n)\sum\limits_{\substack{k\\ p_k<p^*}} n_k[(C/L)-\bar{d}_k] & \text{if} \quad p^c \geqslant C/L \\ (1/n)\sum\limits_{\substack{k\\ p_k\geqslant p^*}} n_k[\bar{d}_k-(C/L)] & \text{if} \quad p^c < C/L.\end{cases} \tag{6}$$

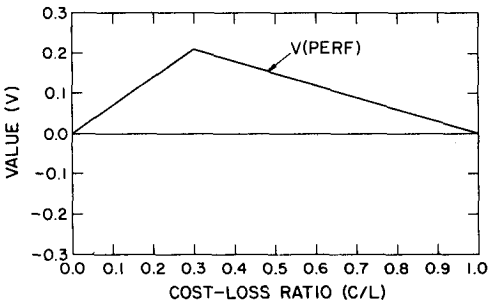

FIG. 2. The basic value diagram in the cost-loss ratio situation, when the sample climatological probability p^c is equal to 0.3.

$V(\text{CAT})$ is a linear function of C/L, with a positive slope equal to

$$(1/n)\sum_{\substack{k\\ p_k<p^*}} n_k \quad \text{for} \quad p^c>C/L$$

and a negative slope equal to

$$(1/n)\sum_{\substack{k\\ p_k\geqslant p^*}} n_k \quad \text{for} \quad p^c<C/L.$$

Thus, $V(\text{CAT})$ attains relative minima at $C/L=0$ and 1. Specifically,

$$V(\text{CAT})=-(1/n)\sum_{\substack{k\\ p_k<p^*}} n_k\bar{d}_k \quad \text{when} \quad C/L=0$$

and

$$V(\text{CAT})=-(1/n)\sum_{\substack{k\\ p_k\geqslant p^*}} n_k(1-\bar{d}_k) \quad \text{when} \quad C/L=1.$$

Note that these minima are, in general, negative.

$V(\text{CAT})$ attains its maximum value at $C/L=p^c$, and this value, $V(\text{CAT})_{\text{MAX}}$, is, from (6),

$$V(\text{CAT})_{\text{MAX}}=(1/n)\sum_{\substack{k\\ p_k<p^*}} n_k(p^c-\bar{d}_k). \tag{7}$$

$V(\text{CAT})_{\text{MAX}}$ can be positive, negative or zero. Specifically, $V(\text{CAT})_{\text{MAX}}$ is positive (negative, zero) when

$$p^c>(<,=)\sum_{\substack{k\\ p_k<p^*}} n_k\bar{d}_k/\sum_{\substack{k\\ p_k<p^*}} n_k.$$

If $V(\text{CAT})_{\text{MAX}}$ is negative, then the expense associated with categorical forecasts is greater than the expense associated with climatological forecasts for all values of C/L. In general, however, $V(\text{CAT})_{\text{MAX}}$ will be negative only for very inaccurate sets of categorical forecasts.[12] $V(\text{CAT})_{\text{MAX}}$ will be positive for most sets of forecasts, and, when $V(\text{CAT})_{\text{MAX}}$ is positive, $V(\text{CAT})=0$ when

$$C/L=\sum_{\substack{k\\ p_k<p^*}} n_k\bar{d}_k/\sum_{\substack{k\\ p_k<p^*}} n_k \quad \text{for} \quad p^c>C/L$$

and when

$$C/L=\sum_{\substack{k\\ p_k\geqslant p^*}} n_k\bar{d}_k/\sum_{\substack{k\\ p_k\geqslant p^*}} n_k \quad \text{for} \quad p^c<C/L.$$

Thus, even when $V(\text{CAT})_{\text{MAX}}$ is positive, $V(\text{CAT})$ is

[12] If the set of n categorical forecasts of concern consists of n_{11} occasions on which adverse weather was both forecast and observed, n_{12} occasions on which adverse weather was forecast but not observed, n_{21} occasions on which adverse weather was observed but not forecast, and n_{22} occasions on which adverse weather was neither observed nor forecast ($n_{11}+n_{12}+n_{21}+n_{22}=n$), then $V(\text{CAT})_{\text{MAX}}$ is positive (negative, zero) if $n_{11}n_{22}>(<,=)n_{12}n_{21}$.

negative over the two half-open intervals

$$[0, \sum_{\substack{k \\ p_k < p^*}} n_k \bar{d}_k / \sum_{\substack{k \\ p_k < p^*}} n_k)$$

and

$$(\sum_{\substack{k \\ p_k \geqslant p^*}} n_k \bar{d}_k / \sum_{\substack{k \\ p_k \geqslant p^*}} n_k, 1].$$

The fact that $V(\text{CAT})$ is negative for some values of C/L for *all* sets of categorical forecasts (except perfect forecasts) was first demonstrated by Thompson (1952).

It may also be of interest to determine the value of categorical forecasts relative to the value of perfect forecasts. If we define the relative value of categorical forecasts, $RV(\text{CAT})$, as the fraction of $V(\text{PERF})$ which is realized with the set of categorical forecasts of concern, then

$$RV(\text{CAT}) = V(\text{CAT})/V(\text{PERF}). \tag{8}$$

The range of $RV(\text{CAT})$ is the half-open interval $(-\infty, 1]$. Specifically, $RV(\text{CAT})=1$ when the categorical forecasts are perfect, while $RV(\text{CAT})=-\infty$ when $C/L=0$ or 1. Moreover, $RV(\text{CAT})$ is negative when $E(\text{CAT})>E(\text{CLIM})$.

3) PROBABILISTIC FORECASTS

The value of probabilistic forecasts, $V(\text{PROB})$, is, from (1) and (4),

$$V(\text{PROB}) = \begin{cases} (1/n) \sum_{\substack{k \\ p_k < C/L}} n_k[(C/L) - \bar{d}_k] & \text{if } p^c \geqslant C/L \\ (1/n) \sum_{\substack{k \\ p_k \geqslant C/L}} n_k[\bar{d}_k - (C/L)] & \text{if } p^c < C/L. \end{cases} \tag{9}$$

$V(\text{PROB})$ is a nonlinear function of C/L (see Footnote 11), with a positive slope when $C/L<p^c$ and a negative slope when $C/L>p^c$. Thus, as in the case of $V(\text{CAT})$, $V(\text{PROB})$ attains relative minima at $C/L=0$ and 1. $V(\text{PROB})$ also attains its maximum value at $C/L=p^c$, and this value, $V(\text{PROB})_{\text{MAX}}$, is, from (9),

$$V(\text{PROB})_{\text{MAX}} = (1/n) \sum_{\substack{k \\ p_k < p^c}} n_k(p^c - \bar{d}_k). \tag{10}$$

$V(\text{PROB})_{\text{MAX}}$ is postive (negative, zero) when

$$p^c > (<, =) \sum_{\substack{k \\ p_k < p^c}} n_k \bar{d}_k / \sum_{\substack{k \\ p_k < p^c}} n_k.$$

However, unless the set of probabilistic forecasts of concern is extremely unreliable, $V(\text{PROB})_{\text{MAX}}$ will be positive.[13].

Now, a set of probabilistic forecasts is either reliable or unreliable. If the forecasts are reliable, then $p_k = \bar{d}_k$ for all k ($k=1, \ldots, K$). In this case, $V(\text{PROB})$ in (9) is greater than or equal to zero for all values of C/L, with equality only if $C/L=0$ or 1. Thus, *the value (expense) associated with reliable probabilistic forecasts is, in general, greater (less) than the value (expense) associated with climatological forecasts for all activities or operations.* On the other hand, if the forecasts are unreliable, then $p_k \neq \bar{d}_k$ for some k ($k=1, \ldots, K$). In this case, $V(\text{PROB})$ can be positive, negative or zero. Specifically, $V(\text{PROB})$ is positive (negative, zero) if

$$C/L > (<, =) \sum_{\substack{k \\ p_k < C/L}} n_k \bar{d}_k / \sum_{\substack{k \\ p_k < C/L}} n_k \text{ when } 0 \leqslant C/L \leqslant p^c$$

and if

$$C/L < (>, =) \sum_{\substack{k \\ p_k \geqslant C/L}} n_k \bar{d}_k / \sum_{\substack{k \\ p_k \geqslant C/L}} n_k \text{ when } p^c < C/L \leqslant 1.$$

We determine $V(\text{PROB})$ [or, equivalently, $E(\text{PROB})$ and $E(\text{CLIM})$] as a function of C/L, for several sets of unreliable forecasts in Section 4.

The relative value of probabilistic forecasts can be defined in a manner similar to that used to define the relative value of categorical forecasts. Specifically, if we denote the ratio of concern by $RV(\text{PROB})$, then

$$RV(\text{PROB}) = V(\text{PROB})/V(\text{PERF}). \tag{11}$$

The range of $RV(\text{PROB})$ is also, in general, the half-open interval $(-\infty, 1]$.[14] Moreover, $RV(\text{PROB})$ is negative when $E(\text{PROB})$ is greater than $E(\text{CLIM})$.

d. Comparison of categorical and probabilistic forecasts

It is obviously of some interest to compare the value of categorical and probabilistic forecasts. Such a comparison can be accomplished simply by taking the difference between $V(\text{PROB})$ and $V(\text{CAT})$, or equivalently between $E(\text{CAT})$ and $E(\text{PROB})$, since $V(\text{PROB}) - V(\text{CAT}) = E(\text{CAT}) - E(\text{PROB})$. Thus, from (3) and (4),

$$V(\text{PROB}) - V(\text{CAT}) = \begin{cases} (1/n) \sum_{\substack{k \\ C/L \leqslant p_k < p^*}} n_k[\bar{d}_k - (C/L)] & \text{if } C/L < p^* \\ 0 & \text{if } C/L = p^* \\ (1/n) \sum_{\substack{k \\ p^* \leqslant p_k < C/L}} n_k[(C/L) - \bar{d}_k] & \text{if } C/L > p^*. \end{cases} \tag{12}$$

This difference is, of course, a nonlinear function of C/L (see Footnote 11).

[13] Of course, $V(\text{PROB})_{\text{MAX}}$, in (10), is positive if the forecasts are reliable (i.e., if $p_k = \bar{d}_k$ for all k; $k=1, \ldots, K$).

[14] When the forecasts are reliable, $RV(\text{PROB})$ approaches $n_1/[n(1-p^c)]$ as C/L approaches zero and $RV(\text{PROB})$ approaches n_K/np^c as C/L approaches one, where n_1 and n_K are the number of forecasts for which $p=p_1=0$ and $p=p_K=1$, respectively.

TABLE 2. Two hypothetical samples of reliable probabilistic forecasts, in which the forecasts are uniformly (Sample 1) and nonuniformly (Sample 2) distributed over the standard set of 11 equally spaced probability values.

		Sample 1		Sample 2	
Class number k	Probability value p_k	Number of forecasts n_k	Observed relative frequency d_k	Number of forecasts n_k	Observed relative frequency d_k
1	0.0	10	0.0	20	0.0
2	0.1	10	0.1	30	0.1
3	0.2	10	0.2	30	0.2
4	0.3	10	0.3	20	0.3
5	0.4	10	0.4	15	0.4
6	0.5	10	0.5	16	0.5
7	0.6	10	0.6	15	0.6
8	0.7	10	0.7	20	0.7
9	0.8	10	0.8	15	0.8
10	0.9	10	0.9	10	0.9
11	1.0	10	1.0	9	1.0
Total/average		$110(=n)$	$0.50(=p^c)$	$200(=n)$	$0.41(=p^c)$

Now, if the probabilistic forecasts of concern are reliable (i.e., if $p_k=\bar{d}_k$ for all k), then $V(\text{PROB})-V(\text{CAT})$ in (12) is greater than or equal to zero for all values of C/L, with equality (in general) only if $C/L=p^*$.[15] Thus, *the value of reliable probabilistic forecasts is, in general, greater than the value of categorical forecasts for all activities or operations.* On the other hand, if the forecasts are unreliable (i.e., if $p_k\neq\bar{d}_k$ for some k), then $V(\text{PROB})-V(\text{CAT})$ can be positive, negative or zero. Specifically, this difference is positive (negative, zero) if

$$C/L<(>,=)\sum_{\substack{k\\ C/L\leqslant p_k<p^*}} n_k\bar{d}_k\Big/\sum_{\substack{k\\ C/L\leqslant p_k<p^*}} n_k \quad \text{when} \quad C/L<p^*$$

and if

$$C/L>(<,=)\sum_{\substack{k\\ p^*\leqslant p_k<C/L}} n_k\bar{d}_k\Big/\sum_{\substack{k\\ p^*\leqslant p_k<C/L}} n_k \quad \text{when} \quad C/L>p^*.$$

We examine this difference [or equivalently $E(\text{CAT})-E(\text{PROB})$], as a function of C/L, for several sets of unreliable probabilistic forecasts in Section 4.

The difference in value between probabilistic forecasts and categorical forecasts, relative to the difference in value between perfect forecasts and categorical forecasts, may also be of interest. If we denote this ratio by $RV(\text{PROB/CAT})$, then

$$RV(\text{PROB/CAT})=[V(\text{PROB})-V\text{CAT})]/\times[V(\text{PERF})-V(\text{CAT})], \quad (13)$$

or, from (8) and (11),

$$RV(\text{PROB/CAT})=[RV(\text{PROB})-RV(\text{CAT})]/\times[1-RV(\text{CAT})]. \quad (14)$$

The total range of $RV(\text{PROB/CAT})$ is the half-open interval $(-\infty, 1]$, while the range of $RV(\text{PROB/CAT})$ for reliable probabilistic forecasts is the closed unit interval $[0, 1]$. $RV(\text{PROB/CAT})=1$ when $V(\text{PROB})=V(\text{PERF})$, and it is positive (negative) when the value of probabilistic (categorical) forecasts exceeds the value of categorical (probabilistic) forecasts.

4. Value of climatological, categorical and probabilistic forecasts: Some hypothetical and real sets of forecasts

Some expressions for and relationships between the expense and/or value associated with the use (by decision makers) of climatological, categorical, probabilistic and perfect forecasts were derived in Section 3. In particular, we demonstrated that the value (expense) for reliable probabilistic forecasts is greater (less) than or equal to the value (expense) for climatological and categorical forecasts. These and other results and relationships will be illustrated in this section by computing the expenses associated with these different types of forecasts for both hypothetical and real sets of forecasts.

a. Reliable probabilistic forecasts

We first consider sets of reliable probabilistic forecasts, and since real samples of forecasts are generally not (completely) reliable, we shall be concerned with hypothetical sets of forecasts in this subsection. Two such sets of forecasts are presented in Table 2. The first set (Sample 1) consists of 110 forecasts uniformly

[15] Since $E(\text{PROB})$ is a piecewise linear function of C/L (see Footnote 11), $V(\text{PROB})-V(\text{CAT})$ is actually equal to zero for all values of C/L in the interval from the largest p_k less than p^* to the smallest p_k greater than or equal to p^*.

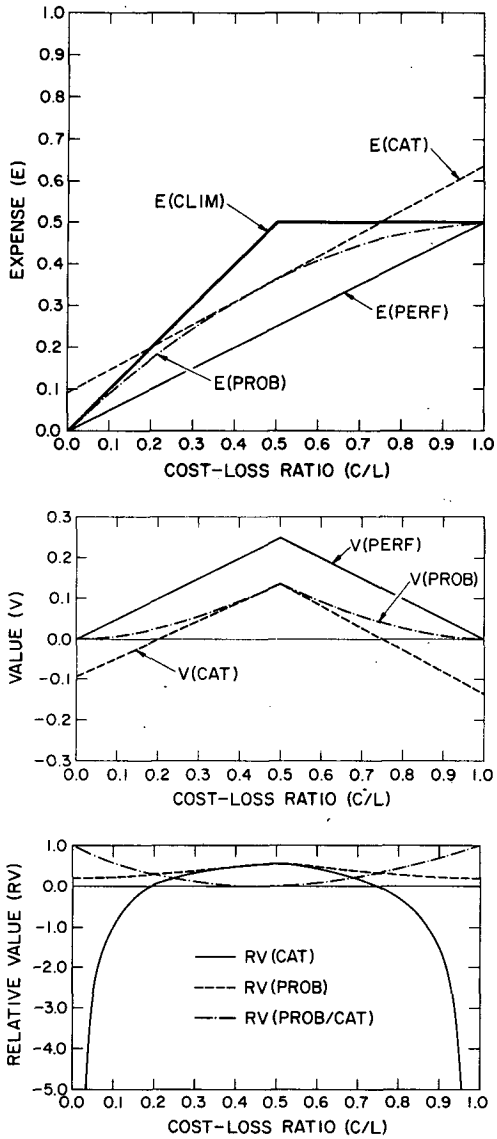

FIG. 3. (a) The expense diagram for the first set of forecasts (Sample 1) presented in Table 2, with $p^*=0.5$; (b) the value diagram for Sample 1 ($p^*=0.5$); and (c) the relative value diagram for Sample 1 ($p^*=0.5$).

distributed over the "standard" set of 11 equally spaced probability values. The sample climatological probability p^c for this set of forecasts is 0.50 (=55/110). The expense and value diagrams for Sample 1, when $p^*=0.5$, are depicted in Figs. 3a and 3b, respectively. Fig. 3a shows the (standardized) expense associated with climatological, categorical, probabilistic and perfect forecasts as a function of C/L, while Fig. 3b describes the value of categorical, probabilistic and perfect forecasts also as a function of C/L.

With regard to the expenses associated with (value of) climatological and categorical forecasts, note that $E(\text{CLIM})\geqslant E(\text{CAT})$ [i.e., $V(\text{CAT})\geqslant 0$] for $0.20\leqslant C/L\leqslant 0.75$ and $E(\text{CLIM})<E(\text{CAT})$ [i.e., $V(\text{CAT})<0$] for $0.00\leqslant C/L<0.20$ and for $0.75<C/L\leqslant 1.00$ (see Figs. 3a and 3b and Section 3c). That is, the expense (value) for categorical forecasts is less (greater) than the expense (value) for climatological forecasts for a central range of values of C/L, while the expense (value) for categorical forecasts is greater (less) than the expense (value) for climatological forecasts for values of C/L outside of this interval. Further, the minimum values of $V(\text{CAT})$ are -0.091 at $C/L=0$ and -0.136 at $C/L=1$, while the maximum value of $V(\text{CAT})$, $V(\text{CAT})_{\text{MAX}}$, is 0.136, which occurs at $C/L=p^c=0.50$ [see Eq. (7)].

Since the forecasts in Sample 1 are reliable, $E(\text{CLIM})\geqslant E(\text{PROB})$ [i.e., $V(\text{PROB})\geqslant 0$] and $E(\text{CAT})\geqslant E(\text{PROB})$ [i.e., $V(\text{PROB})\geqslant V(\text{CAT})$] for all values of C/L (see Sections 3c and 3d). That is, the expense (value) for probabilistic forecasts is less (greater) than or equal to the expense (value) for climatological and categorical forecasts for all activities or operations. In this regard, $V(\text{PROB})=0$ only for $C/L=0$ and 1 and $V(\text{PROB})=V(\text{CAT})$ only for $0.40\leqslant C/L\leqslant 0.50$ (see Footnote 15). Moreover, the expense (value) for probabilistic forecasts is less (greater) than the expense (value) for climatological and categorical forecasts—i.e., $E(\text{PROB})[V(\text{PROB})]<(>)\min[E(\text{CLIM}), E(\text{CAT})]$ $\{\max[0,V(\text{CAT})]\}$—by an appreciable amount over significant ranges of values of C/L (specifically, for $0.1\lesssim C/L\lesssim 0.3$ and for $0.6\lesssim C/L\lesssim 0.9$).[16] Finally, the maximum value of $V(\text{PROB})$, $V(\text{PROB})_{\text{MAX}}$, is also 0.136, which occurs at $C/L=p^c=0.50$ [see Eq. (10)].

A "relative value diagram" for this set of forecasts is depicted in Fig. 3c. As expected, $RV(\text{CAT})$ is positive for $0.20<C/L<0.75$ and negative for all values of C/L outside of this interval (it actually approaches minus infinity as C/L approaches zero or one). On the other hand, $RV(\text{PROB})$ and $RV(\text{PROB/CAT})$ are non-negative for all values of C/L. $RV(\text{CAT})$ and $RV(\text{PROB})$ both attain a maximum value of 0.544 (=0.136/0.250) at $C/L=p^c=0.50$, while $RV(\text{PROB/CAT})$ attains a maximum value of one at $C/L=0$ and 1. Finally, $RV(\text{PROB})$ approaches a minimum value of 0.182 when C/L approaches zero or one (see footnote 14).

The effect of selecting a different value of p^*, the critical probability value used to transform probabilistic forecasts into categorical forecasts, is indicated in Fig. 4, in which expense diagrams for Sample 1 are presented for $p^*=0.3$ (Fig. 4a) and for $p^*=0.7$ (Fig. 4b). First, it should be noted that changing the value of p^* changes the slope of the line representing $E(\text{CAT})$ [however, it does not affect the "curves" representing $E(\text{CLIM})$, $E(\text{PROB})$ and $E(\text{PROB})$]. For reliable forecasts, this line is, in general, tangent to the curve representing $E(\text{PROB})$ at the point $C/L=p^*$ [actually, $E(\text{CAT})$ may be tangent to $E(\text{PERF})$ over a set of

[16] The symbol $\lesssim$ denotes an "inexact inequality," and it is used in this paper to describe approximate intervals over which certain relationships and/or results hold.

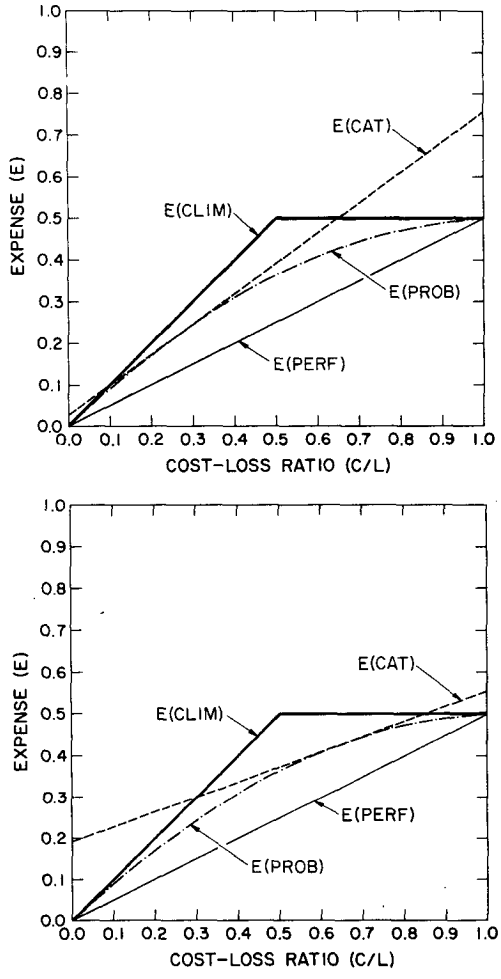

FIG. 4. The expense diagrams for Sample 1 (Table 2), with (a) $p^*=0.3$ and (b) $p^*=0.7$.

values of C/L in the vicinity of p^*; see Footnote 15]. Thus, while E(CAT) is tangent to E(PROB) at $C/L=p^*=0.5$ in Fig. 3a, E(CAT) is tangent to E(PROB) at $C/L=p^*=0.3$ in Fig. 4a and at $C/L=p^*=0.7$ in Fig. 4b. Specifically, the effect of selecting a different value of p^* is to decrease E(CAT) for some values of C/L and to increase E(CAT) for other values of C/L, and these changes, in turn, lead to increases (decreases) and decreases (increases) in V(CAT) [V(PROB) $-V$(CAT)], respectively. When $p^*=0.3$ (Fig. 4a), E(CAT) decreases appreciably for $C/L\lesssim 0.30$ and increases appreciably for $C/L\gtrsim 0.50$, *vis-à-vis* the situation in which $p^*=0.5$ (cf. Fig. 3a). Thus, for example, V(PROB)$-V$(CAT) decreases for $C/L\lesssim 0.30$ and increases for $C/L\gtrsim 0.50$. Moreover, Fig. 4a indicates that, in this case, appreciable differences between E(PROB) and min [E(CLIM),E(CAT)] are realized for $0.4\lesssim C/L\lesssim 0.9$. When $p^*=0.7$ (Fig. 4b), the situation is reversed; that is, E(CAT)$-E$(PROB) is smaller for $C/L\gtrsim 0.7$ and larger for $C/L\lesssim 0.5$, relative to the situation in which $p^*=0.5$ (cf. Fig. 3a). In this case, appreciable differences between E(PROB) and min [E(CLIM),E(CAT)] are obtained for $0.1\lesssim C/L\lesssim 0.5$ and for $0.8\lesssim C/L\lesssim 0.9$. Thus, the selection of a particular value of p* can substantially decrease the positive difference between the expense associated with categorical forecasts and the expense associated with reliable probabilistic forecasts for some values of C/L (*viz.*, for those values of C/L in the vicinity of p^*), but at the same time this choice of p* will necessarily lead to increases in this difference for other values of C/L.

The second set of forecasts presented in Table 2 (Sample 2) consists of 200 forecasts *nonuniformly* distributed over the same set of 11 equally spaced probability values ($p^c=0.41$ for this set of forecasts). This hypothetical sample of reliable forecasts is considered here because the distributions of forecasts found in real sets of forecasts more closely approximate the distribution of forecasts in Sample 2 than the uniform distribution in Sample 1 (see Section 4b). The expense diagram for this set of forecasts, when $p^*=0.5$, is depicted in Fig. 5 (to conserve space, the value and relative value diagrams for Sample 2 are not presented). As expected (since the forecasts are reliable), E(PROB) is less than or equal to E(CLIM) and E(CAT) for all values of C/L. Moreover, E(PROB) is appreciably less than min[E(CLIM),E(CAT)] for $0.1\lesssim C/L\lesssim 0.3$ and for $0.6\lesssim C/L\lesssim 0.9$. A comparison of Fig. 5 and Fig. 3a reveals that these expense diagrams are very similar despite the fact that the number and distribution of forecasts in the two samples are quite different.

b. Unreliable probabilistic forecasts

1) HYPOTHETICAL SAMPLES

Two hypothetical sets of unreliable probabilistic forecasts are presented in Table 3. These samples both consist of 110 forecasts uniformly distributed over the

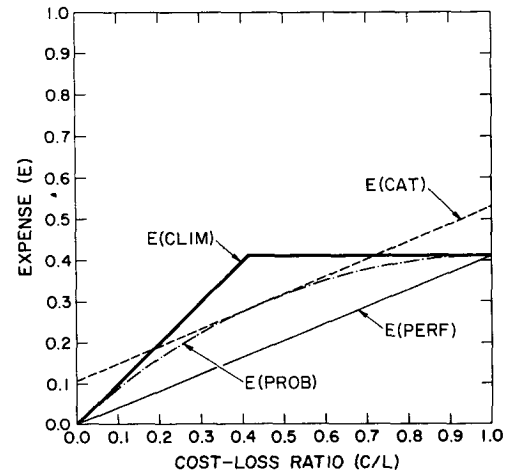

FIG. 5. The expense diagram for the second set of forecasts (Sample 2) presented in Table 2, with $p^*=0.5$.

TABLE 3. Two hypothetical samples of unreliable probabilistic forecasts, in which the forecasts are uniformly distributed over the standard set of 11 equally spaced probability values. Sample 3 is moderately unreliable, while Sample 4 is extremely unreliable.

		Sample 3		Sample 4	
Class number k	Probability value p_k	Number of forecasts n_k	Observed relative frequency $\bar{d}_k$	Number of forecasts n_k	Observed relative frequency $\bar{d}_k$
1	0.0	10	0.1	10	0.5
2	0.1	10	0.2	10	0.5
3	0.2	10	0.3	10	0.5
4	0.3	10	0.4	10	0.5
5	0.4	10	0.5	10	0.5
6	0.5	10	0.5	10	0.5
7	0.6	10	0.5	10	0.5
8	0.7	10	0.6	10	0.5
9	0.8	10	0.7	10	0.5
10	0.9	10	0.8	10	0.5
11	1.0	10	0.9	10	0.5
Total/average		$110(=n)$	$0.5(=p^c)$	$110(=n)$	$0.5(=p^c)$

standard set of 11 equally spaced probability values ($p^c=0.50$ for both sets of forecasts). The first sample in this table (Sample 3) is similar to Sample 1 in Table 2, except that $p_k=\bar{d}_k-0.1(k=1, \ldots, 5)$, $p_k=\bar{d}_k(k=6)$, and $p_k=\bar{d}_k+0.1(k=7, \ldots, 11)$ in Sample 3, whereas $p_k=\bar{d}_k$ for all k in Sample 1. We shall refer to Sample 3 as "moderately unreliable." The expense and value diagrams for this set of forecasts, when $p^*=0.5$, are depicted in Figs. 6a and 6b, respectively. First, note that $E(\text{CLIM})\geqslant E(\text{PROB})$ [i.e., $V(\text{PROB})\geqslant 0$] for all except very small and very large values of C/L. Specifically, $V(\text{PROB})$ is slightly negative for $0.0<C/L\lesssim 0.1$ and for $0.90\lesssim C/L<1.0$. With regard to the expense (value) associated with categorical and probabilistic forecasts, $E(\text{CAT})\geqslant E(\text{PROB})$ [i.e., $V(\text{PROB})\geqslant V(\text{CAT})$] for all values of C/L, with equality only for $0.40<C/L\leqslant 0.50$. These (and other) relationships are also illustrated in the relative value diagram in Fig. 6c. Thus, despite the fact that this set of forecasts is moderately unreliable, the value of probabilistic forecasts exceeds the value of climatological and categorical forecasts for most activities or operations.

The second hypothetical set of forecasts presented in Table 3 (Sample 4) will be referred to as "extremely unreliable," because the observed relative frequency $\bar{d}_k$ is equal to 0.50 for all k ($k=1, \ldots, 11$). That is, for Sample 4, adverse weather occurred on exactly one-half of the 10 forecasting occasions associated with each of the 11 probability values (p_k; $k=1, \ldots, 11$). The expense diagram for this set of forecasts, when $p^*=0.5$, is depicted in Fig. 7. Note that $E(\text{CLIM})\leqslant E(\text{PROB})\leqslant E(\text{CAT})$ for all values of C/L. Thus, for all activities or operations, the expense for climatological forecasts is less than or equal to the expense for categorical and probabilistic forecasts and, in addition, the expense associated with probabilistic forecasts is less than the expense associated with categorical forecasts. Thus, decision makers, in general, would prefer probabilistic forecasts to categorical forecasts, but they would prefer climatological forecasts to both categorical and probabilistic forecasts. It should be emphasized that since the forecasts in Sample 4 are extremely unreliable, such forecasts are not representative of probabilistic forecasts prepared on either an operational or an experimental basis in meteorology in recent years (see, e.g., Murphy and Winkler, 1977).

2) REAL SAMPLES

A set of operational probability of precipitation (PoP) forecasts for Chicago, Ill., is presented in Table 4. This sample consists of all of the PoP forecasts formulated by the National Weather Service (NWS) forecasters at Chicago during the four-year period from July 1972–June 1976. The reliability of these forecasts is representative of the reliability of PoP forecasts currently issued to the public by NWS forecasters. Specifically, these forecasts are quite—but not completely—reliable. The expense, value and relative value diagrams for this set of forecasts, when $p^*=0.5$, are depicted in Figs. 8a, 8b and 8c, respectively. Note that, despite the fact that these forecasts are not (completely) reliable, $E(\text{PROB})[V(\text{PROB})]\leqslant(\geqslant)\min[E(\text{CLIM}), E(\text{CAT}]\{\max[0,V(\text{CAT})]\}$ for all values of C/L. Moreover, probabilistic forecasts provide an appreciable increase in value *vis-à-vis* both climatological and categorical forecasts for significant ranges of values of C/L (specifically, for $0.05\lesssim C/L\lesssim 0.30$ and for $0.50\lesssim C/L\lesssim 0.75$). It should also be mentioned that the expense, value and relative value diagrams corresponding to specific subsets of this set of forecasts (for example, subsets of forecasts formulated by individual forecasters) are very similar to the diagrams presented in Fig. 8.

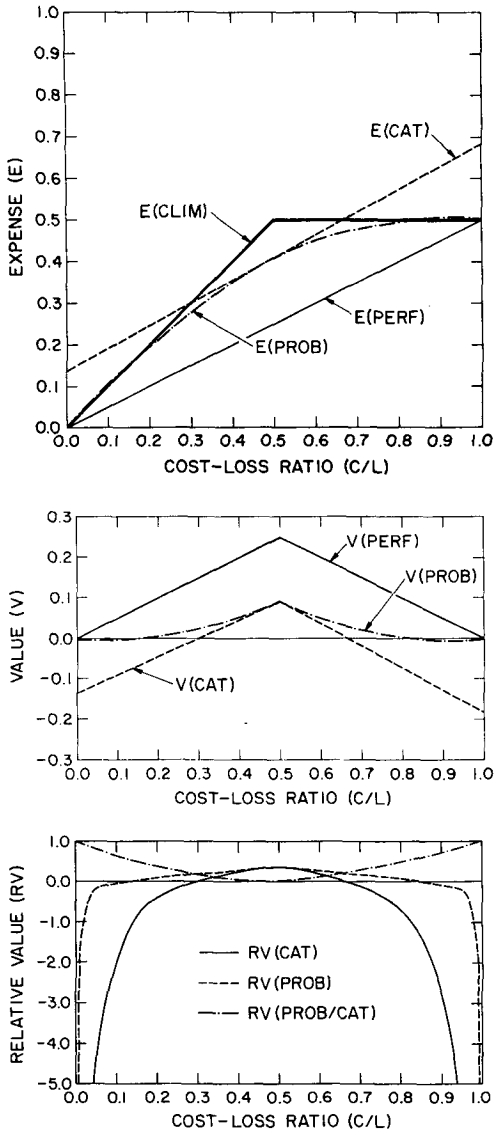

FIG. 6. (a) The expense diagram for the first set of forecasts (Sample 3) presented in Table 3, with $p^*=0.5$; (b) the value diagram for Sample 3 ($p^*=0.5$); and (c) the relative value diagram for Sample 3 ($p^*=0.5$).

The effect of using a different value of p^* to transform the probabilistic forecasts into categorical forecasts for this sample of PoP forecasts is indicated in Fig. 9, in which expense diagrams are presented for $p^*=0.3$ (Fig. 9a) and $p^*=0.7$ (Fig. 9b). As indicated in Section 4a, the choice of a particular value of p^* has the effect of decreasing the (generally positive) difference between the expense associated with categorical and probabilistic forecasts for values of C/L in the immediate vicinity of p^* and of increasing this difference for other values of C/L. These effects are readily apparent if we compare Figs. 9a and 9b with Fig. 8a.

Finally, a set of probabilistic minimum temperature forecasts formulated by NWS forecasters at Albuquerque, N. M., is presented in Table 5. These forecasts were prepared during a six-week period each spring from 1970–76, first on an experimental basis and then more recently on an operational basis.[17] Since the forecasters have not had very much experience making such forecasts, the forecasts are not as reliable as the PoP forecasts considered previously. The expense value and relative value diagrams for this set of forecasts, when $p^*=0.5$, are presented in Figs. 10a, 10b and 10c, respectively. A comparison of categorical and probabilistic forecasts reveals that $E(\text{CAT})\geqslant E(\text{PROB})$ [i.e., $V(\text{PROB})\geqslant C(\text{CAT})$] for all values of C/L except for a very narrow range of values in the vicinity of $C/L=0.35$. Further, note that $E(\text{CLIM}\geqslant E(\text{PROB})$

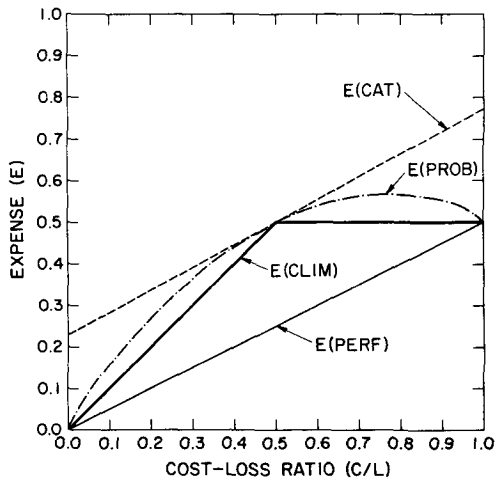

FIG. 7. The expense diagram for the second set of forecasts (Sample 4) presented in Table 3, with $p^*=0.5$.

TABLE 4. A sample of operational precipitation probability forecasts formulated by NWS forecasters at Chicago, Ill., during the period from July 1972–June 1976.

Class number k	Probability value p_k	Number of forecasts n_k	Observed relative frequency $\bar{d}_k$
1	0.00	1134	0.014
2	0.02	816	0.020
3	0.05	2502	0.039
4	0.10	3138	0.086
5	0.20	3206	0.181
6	0.30	1574	0.285
7	0.40	1031	0.353
8	0.50	1277	0.447
9	0.60	587	0.627
10	0.70	1065	0.667
11	0.80	564	0.730
12	0.90	468	0.861
13	1.00	152	0.974
Total/average		17514 ($=n$)	0.251 ($=p^c$)

[17] The forecasts were made for several areas in New Mexico in which deciduous fruits are grown in significant quantities, and they relate to the probability that the minimum temperature will not exceed 28°F. They were first disseminated to orchardists and others on a regular basis in 1975.

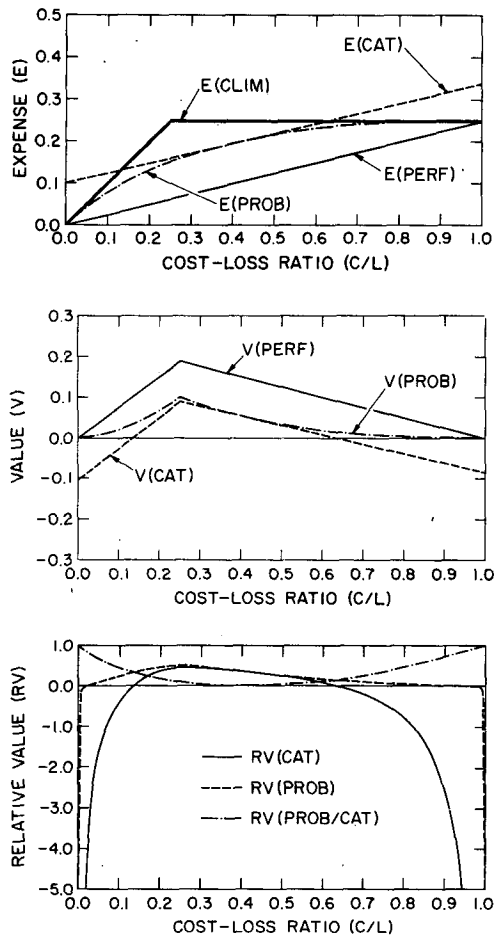

FIG. 8. (a) The expense diagram for the precipitation probability forecasts presented in Table 4, with $p^*=0.5$; (b) the value diagram for the same set of forecasts ($p^*=0.5$); and (c) the relative value diagram for this set of forecasts ($p^*=0.5$).

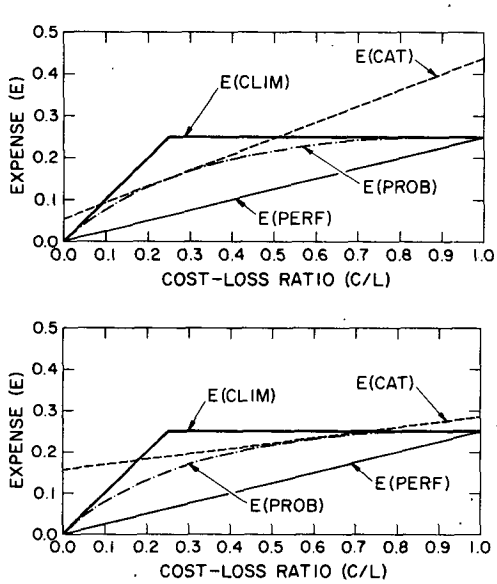

FIG. 9. The expense diagrams for the precipitation probability forecasts presented in Table 4, with (a) $p^*=0.3$ and (b) $p^*=0.7$.

TABLE 5. A sample of experimental probabilistic minimum temperature forecasts formulated by NWS forecasters at Albuquerque, N. M., during the months of April and May from 1970–76.

Class number k	Probability value p_k	Number of forecasts n_k	Observed relative frequency $\bar{d}_k$
1	0.00	524	0.013
2	0.02	14	0.000
3	0.05	45	0.022
4	0.10	161	0.118
5	0.20	115	0.174
6	0.30	76	0.342
7	0.40	57	0.298
8	0.50	56	0.393
9	0.60	84	0.429
10	0.70	79	0.544
11	0.80	81	0.654
12	0.90	79	0.696
13	1.00	72	0.764
Total/average		1443($=n$)	0.245($=p^c$)

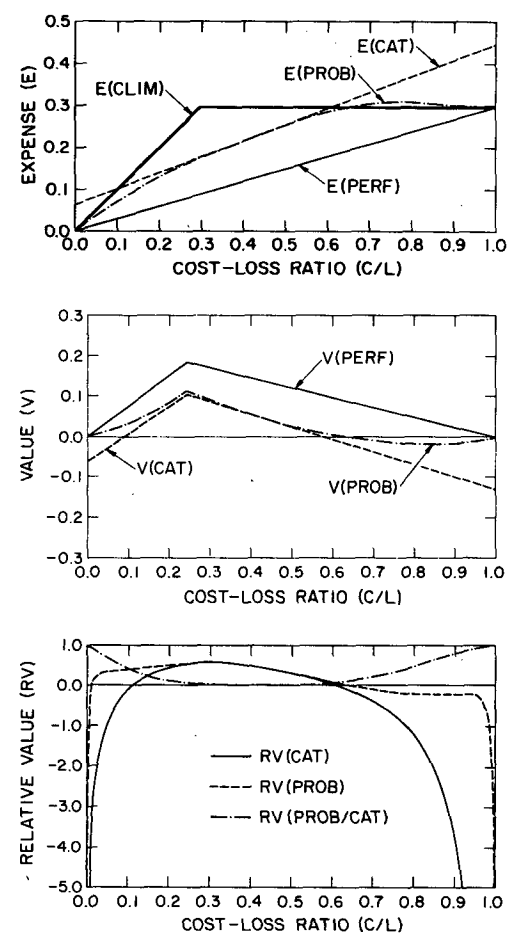

FIG. 10. The expense diagram for the probabilistic minimum temperature forecasts presented in Table 5, with $p^*=0.5$; (b) the value diagram for the same set of forecasts ($p^*=0.5$); and (c) the relative value diagram for this set of forecasts ($p^*=0.5$.)

[i.e., $V(\text{PROB}) \geqslant 0$] for $0.00 \leqslant C/L \lesssim 0.70$, while $E(\text{CLIM}) \leqslant E(\text{PROB})$ [i.e., $V(\text{PROB}) \leqslant 0$] for $0.70 \lesssim C/L \leqslant 1.00$. Specifically, the value of probabilistic forecasts is appreciably greater than the value of climatological forecasts for activities for which $0.05 \lesssim C/L \lesssim 0.65$, while the value of climatological forecasts exceeds the value of probabilistic forecasts for operations for which $0.75 \lesssim C/L \lesssim 0.95$. Thus, despite the fact that the probabilistic forecasts of concern are quite unreliable, many (if not most) decision makers would prefer these forecasts to climatological and categorical forecasts.[18] Moreover, the value of the probabilistic forecasts can be expected to increase still further, *vis-à-vis* the value of climatological and categorical forecasts, as the forecasters become more experienced at making such forecasts.

5. Summary and conclusion

In this paper we have been concerned with the value of climatological, categorical, probabilistic and perfect forecasts within the framework of the cost-loss ratio situation. First, we derived general expressions for the expense associated with the use—by decision makers—of the different types of forecasts. In these expressions, the expense is specified as a function of the cost-loss ratio and the term "expense" refers to the decision makers' costs and/or losses *per unit loss per forecast.* The expense associated with perfect forecasts represents a lower bound on the expense associated with the other types of forecasts. Then, we formulated measures of the value of categorical, probabilistic and perfect forecasts, in which the value of categorical (probabilistic, perfect) forecasts is defined as the difference between the expense associated with climatological forecasts and the expense associated with categorical (probabilistic, perfect) forecasts. The value of perfect forecasts represents an upper bound on the value of categorical and probabilistic forecasts. Finally, expressions for the relative value of categorical and probabilistic forecasts were defined, in which the relative value of categorical (probabilistic) forecasts is the ratio of the value of categorical (probabilistic) forecasts to the value of perfect forecasts. We also formulated a measure of the value of probabilistic forecasts relative to the value of categorical forecasts.

The expressions for the expense and value associated with these different types of forecasts were examined and compared for situations in which the basic probabilistic forecasts[19] were assumed to be 1) (completely) reliable and 2) unreliable. Specifically, we showed that, *when the forecasts are reliable, the expense (value) associated with probabilistic forecasts is less (greater) than or equal to the expense (value) associated with climatological and categorical forecasts for all values of the cost-loss ratio* (i.e., for all activities or operations).[20] On the other hand, when the forecasts are unreliable, the expense (value) associated with probabilistic forecasts *may* be greater (less) than the expense (value) associated with climatological and/or categorical forecasts for some values of the cost-loss ratio. With regard to the latter, the conditions under which decision makers would prefer climatological and/or categorical forecasts to unreliable probabilistic forecasts were briefly described.

The relationships between the expense and/or value associated with climatological, categorical, probabilistic and perfect forecasts were illustrated by examining several samples of forecasts. Specifically, we considered hypothetical sets of reliable probabilistic forecasts and both hypothetical and real sets of unreliable probabilistic forecasts. The latter included forecasts of precipitation occurrence and minimum temperature formulated by NWS forecasters. The examination of these and other samples of unreliable probabilistic forecasts has indicated that the relationships between the expense (and value) of *reliable* probabilistic forecasts and the expense (and value) of climatological and categorical forecasts are quite robust, in the sense that these relationships appear to hold for most (if not all) values of the cost-loss ratio even for moderately unreliable forecasts.[21] Moreover, these empirical studies have suggested that, when the forecasts are unreliable, the value of probabilistic forecasts is more likely to be exceeded by the value of climatological forecasts than by the value of categorical forecasts.

The results presented in this paper have some implications for operational forecasting procedures and practices. The most important implication, of course, relates to the fact that the value of reliable—and even moderately unreliable—probabilistic forecasts generally exceeds the value of climatological and categorical forecasts. This result suggests that the value of day-to-day weather forecasts could be significantly increased, within the context of many decision-making situations, if such forecasts were routinely expressed in probabilistic terms *and* disseminated to decision makers (including the general public). In this regard, it should be empha-

[18] In this regard, the available data indicate that the cost-loss ratios of orchardists in New Mexico range from approximately 0.02–0.05 (G. T. Gregg, personal communication). Thus, the cost-loss ratios of the decision makers of concern fall in an interval in which the value of these experimental probabilistic forecasts exceeds the value of both climatological and categorical forecasts (see Fig. 10).

[19] The climatological, categorical and perfect forecasts are derived from the probabilistic forecasts and the relevant observations (see Section 3a).

[20] Actually, $E(\text{PROB}) = E(\text{CLIM})$ only for $C/L = 0$ and 1 and $E(\text{PROB}) = E(\text{CAT})$ only for values of C/L in the immediate vicinity of p^* (see Footnote 15).

[21] Many hypothetical and real sets of probabilistic forecasts have been examined in this study, only six of which are considered in Section 4.

sized that the benefits which could be expected from such a probability forecasting program do *not* depend upon scientific advances in the state of the art of weather forecasting (i.e., upon improved numerical or numerical-statistical models or upon improvements in the ability of forecasters to formulate accurate forecasts). If such a program is initiated, an experimental approach should be adopted to both the formulation and dissemination of the forecasts, and the program should be accompanied by a concerted effort to educate decision makers and others regarding the proper interpretation and use of probabilistic forecasts.

Another implication of these results relates to the relative importance of reliability and other inferential attributes of probabilistic forecasts [inferential attributes are concerned with the correspondence between forecasts and observations (see Murphy and Winkler, 1971)]. Reliability has generally been considered to be an important attribute of such forecasts primarily (if not exclusively) because it provides a measure of the forecasters' ability to quantify their uncertainty in probabilistic terms (see, e.g., Winkler and Murphy, 1968; Murphy and Winkler, 1970, 1971). The fact that the value of reliable probabilistic forecasts, in general, exceeds the value of climatological and categorical forecasts provides a different, but equally important, justification for assigning a relatively high priority to this attribute of the forecasts. Of course, since the results relating to the value of *reliable* probabilistic forecasts are quite robust, such forecasts need not be completely reliable in order that their value exceed the value of climatological and categorical forecasts for most (if not all) activities or operations. On the other hand, as probabilistic forecasts become more reliable, these ordinal relationships (on expense and value) are more likely to hold for *all* values of the cost-loss ratio.

In conclusion, this paper has described the results of a study of the value of several different types of weather forecasts in a particular decision-making situation, the cost-loss ratio situation. In view of the importance of this topic, it would seem to be desirable to extend the study to include other types of forecasts (e.g., persistence forecasts) and to investigate the value of all of these forecasts within the framework of other decision-making situations.[22] In addition, the relationships between various inferential attributes of such forecasts and their value are in need of further study.

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[22] It seems reasonable to conjecture that the value of (completely) reliable probabilistic forecasts will be greater than or equal to the value of climatological and categorical forecasts in *all* decision-making situations.

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