

Scalar and Vector Partitions of the Probability Score : Part II. N -State Situation

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ABSTRACT

Scalar and vector partitions of the probability score (PS) in N -state ($N > 2$) situations are described and compared. In N -state, as well as in two-state ($N = 2$), situations these partitions provide similar, but not equivalent (i.e., linearly related), measures of the reliability and resolution of probability forecasts. Specifically, the vector partition, when compared to the scalar partition, decreases the reliability and increases the resolution of the forecasts. A sample collection of forecasts is used to illustrate the differences between these partitions in N -state situations.

Several questions related to the use of scalar and vector partitions of the PS in N -state situations are discussed, including the relative merits of these partitions and the effect upon sample size when forecasts are considered to be vectors rather than scalars. The discussions indicate that the vector partition appears to be more appropriate, in general, than the scalar partition, and that when the forecasts in a collection of forecasts are considered to be vectors rather than scalars the sample size of the collection may be substantially reduced.

1. Introduction

Scalar and vector partitions of the probability score (PS) (Brier, 1950) in the two-state ($N = 2$) situation have recently been described and compared (Murphy, 1972b; hereafter we refer to this paper as Part I). In Part I we demonstrated that these partitions, which are based upon expressions for the PS in which probability forecasts are considered to be scalars and vectors, respectively, provide similar, but not equivalent (i.e., linearly related), measures of the reliability and resolution of the forecasts in the two-state situation. Specifically, we indicated that the vector partition, when compared to the scalar partition, decreases the reliability and increases the resolution of the forecasts.² In addition, we have also recently obtained similar results for the ranked probability score (RPS) (Epstein, 1969), in which the scalar and vector partitions are based upon scalar and vector *cumulative* forecasts, respectively (see Murphy, 1972c).

The purposes of this paper are to describe and compare the scalar and vector partitions of the PS in N -state ($N > 2$) situations and to discuss several questions related to the use of these partitions in such situations. In particular, we extend the results obtained in Part I which relate to the relationship between the scalar and vector partitions to N -state situations and

consider, in some detail, several questions related to the use of these partitions in N -state situations, including the relative merits of these partitions and the effect upon sample size when forecasts are considered to be vectors rather than scalars.³

In Section 2 we briefly describe the differences between scalar and vector forecasts and observations and introduce notation to identify these quantities. The scalar and vector partitions of the PS in N -state situations are presented in Section 3. In Section 4 we compare the scalar and vector partitions and demonstrate that these partitions are not equivalent, i.e., linearly related. A sample collection of forecasts is used to illustrate the differences between these partitions in N -state situations in Section 5. Several questions related to the use of these partitions in N -state situations are discussed in Section 6, including the relative merits of these partitions and the effect upon sample size when forecasts are considered to be vectors rather than scalars. Section 7 consists of a brief summary and conclusion.

2. Scalar and vector forecasts and observations

We assume that the forecasts and observations relate to situations in which the range of the variable of concern has been divided into a set of N mutually exclusive and collectively exhaustive states $\{s_1, \dots, s_N\}$.

We are concerned with collections of forecasts and the relevant observations. When the forecasts and

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² In reality, the scalar and vector partitions are concerned with the reliability and resolution of individual *probabilities* and sets of probabilities, or *forecasts*, respectively (see Section 2).

³ The results described in this paper are applicable in two-state ($N = 2$) as well as in N -state ($N > 2$) situations.

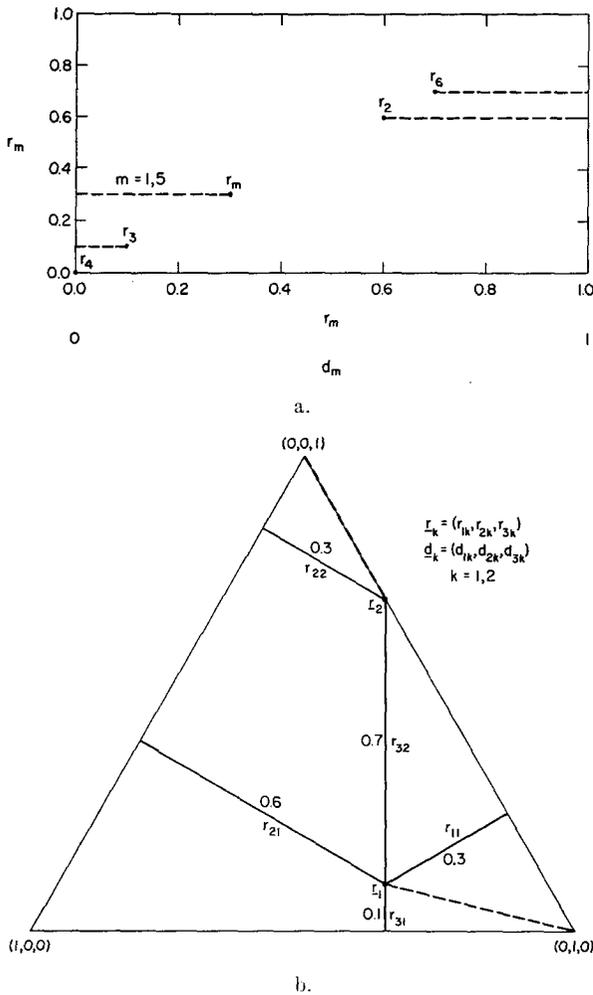


FIG. 1. Two occasions on which the probabilities assigned to a three-state ($N=3$) variable are 0.3, 0.6, 0.1 and 0.0, 0.3, 0.7 respectively, depicted within the appropriate framework when forecasts are considered to be (a) scalars and (b) vectors (the probabilities which correspond to the observed states are underlined). The dashed lines represent the distances between the forecasts and the relevant observations in the respective frameworks. The PS for each forecast equals the square of this distance.

observations are considered to be scalars, we denote the m th forecast in a collection of M scalar forecasts by the scalar r_m ($0 \leq r_m \leq 1$) and the relevant observation by the scalar d_m , where d_m equals one if the state of concern occurs and zero otherwise ($m=1, \dots, M$). When the forecasts and observations are considered to be vectors, we denote the k th forecast in this collection of K ($=M/N$) vector forecasts by the row vector

$$\mathbf{r}_k = (r_{1k}, \dots, r_{Nk}), \quad (r_{nk} \geq 0, \sum_n r_{nk} = 1; n = 1, \dots, N),$$

and the relevant observation by the row vector $\mathbf{d}_k = (d_{1k}, \dots, d_{Nk})$, where d_{nk} equals one if state s_n occurs on the k th occasion and zero otherwise ($n=1, \dots, N$; $k=1, \dots, K$).

The differences between scalar and vector forecasts

in N -state ($N > 2$) situations can be indicated by considering the probabilities assigned to a three-state ($N=3$) variable on two occasions. Suppose that these probabilities are 0.3, 0.6, 0.1 on the first occasion and 0.0, 0.3, 0.7 on the second occasion, and that state s_2 occurs on the first occasion and state s_3 occurs on the second occasion. If we consider forecasts to be scalars, then we have six forecasts ($M=6$): $r_1=0.3$, $r_2=0.6$, $r_3=0.1$, $r_4=0.0$, $r_5=0.3$, $r_6=0.7$. The appropriate framework within which to depict scalar forecasts is the unit line segment (see Part I, p. 274). These forecasts are depicted within this framework in Fig. 1a. If we consider forecasts to be vectors, then we have two forecasts ($K=2$): $\mathbf{r}_1=(0.3, 0.6, 0.1)$ and $\mathbf{r}_2=(0.0, 0.3, 0.7)$. The appropriate framework within which to depict vector forecasts is a regular $(N-1)$ -dimensional simplex, which is an equilateral triangle in the three-state situation (see Pontryagin, 1952, pp. 10-12; see also Part I, p. 274, and Murphy, 1972a). These forecasts are depicted in this framework in Fig. 1b.⁴

The partitions of the PS are based upon the assumption that the probabilities which constitute the forecasts can assume only a finite set of values (see Part I, pp. 274-275). Specifically, we assume that the collection of forecasts of concern consists of M scalar or K ($=M/N$) vector forecasts and that the probabilities can assume only S distinct values. Then, we can identify S distinct scalar forecasts r^s ($s=1, \dots, S$) and, as a result, S distinct subcollections of the collection of M scalar forecasts, where subcollection s consists of the M^s forecasts for which

$$r_m = r^s \quad (m = 1, \dots, M^s; \sum_s M^s = M; s = 1, \dots, S).$$

On the other hand, we can identify T distinct vector forecasts \mathbf{r}^t ($t=1, \dots, T$), where

$$T = \sum_{s=1}^S \binom{N+s-4}{s-1} (S-s+1), \quad (1)$$

in which $\binom{x}{y} = x!/[y!(x-y)!]$ for $0 \leq y \leq x$, $\binom{x}{y} = 1$ for $x = -1$ and $y = 0$, and $\binom{x}{y} = 0$ otherwise. Thus, we can identify T distinct subcollections of the collection of K vector forecasts, where subcollection t consists of the K^t forecasts for which

$$\mathbf{r}_k = \mathbf{r}^t \quad (k = 1, \dots, K^t; \sum_t K^t = K; t = 1, \dots, T),$$

⁴In a regular simplex the natural coordinate system is a "content" coordinate system, in which the perpendicular distances between the point (in the simplex) which represents a vector forecast and the "faces" of the simplex are equal to the components of the forecast. Thus, in the three-state situation, in which a content coordinate system becomes an "area" coordinate system, the distances between the point which represents a vector forecast and the sides of the equilateral triangle are the three components of the forecast (see Springer, 1964, pp. 119-122). The solid lines in Fig. 1b represent the components of the respective forecasts.

in which $\mathbf{r}^t = (r_1, \dots, r_N)^t$.⁵ For these subcollections we denote the relevant scalar observations by d_m^s ($m=1, \dots, M^s$) and the relevant vector observations by \mathbf{d}_k^t ($k=1, \dots, K^t$), where $\mathbf{d}_k^t = (d_{1k}^t, \dots, d_{Nk}^t)$.

3. Scalar and vector partitions: Formulation

The formulation of the scalar and vector partitions of the PS for a collection of M scalar or K ($=M/2$) vector forecasts, respectively, in the two-state ($N=2$) situation has been described in Part I (pp. 275-276). Since these partitions are formulated in the same manner in N -state ($N>2$) situations, we simply reproduce the expressions for these partitions in this paper.

The scalar partition of the PS is $PS(r,d)$, where

$$PS(r,d) = (1/M) \sum_{s=1}^S M^s (r^s - \bar{d}^s)^2 + (1/M) \sum_{s=1}^S M^s \bar{d}^s (1 - \bar{d}^s), \quad (2)$$

in which

$$\bar{d}^s = (1/M^s) \sum_{m=1}^{M^s} d_m^s$$

The terms on the right hand-side (RHS) of (2) are the measures of reliability and resolution, respectively, when the forecasts are considered to be scalars (see Part I, Footnote 10). The range of $PS(r,d)$, in (2), is the closed interval $[0, 2/N]$.

The vector partition of the PS is $PS(\mathbf{r}, \mathbf{d})$, where

$$PS(\mathbf{r}, \mathbf{d}) = (1/K) \sum_{t=1}^T K^t (\mathbf{r}^t - \bar{\mathbf{d}}^t) (\mathbf{r}^t - \bar{\mathbf{d}}^t)' + (1/K) \sum_{t=1}^T K^t \bar{\mathbf{d}}^t (\mathbf{u} - \bar{\mathbf{d}}^t)', \quad (3)$$

in which

$$\bar{\mathbf{d}}^t = (1/K^t) \sum_{k=1}^{K^t} \mathbf{d}_k^t$$

\mathbf{u} is a row vector whose N elements are all equal to one, i.e., $\mathbf{u} = (1, \dots, 1)$, and a prime denotes a column vector. The terms on the RHS of (3) are the measures of reliability and resolution, respectively, when the forecasts are considered to be vectors (see Part I, Footnote 10). The range of $PS(\mathbf{r}, \mathbf{d})$, in (3), is the closed interval $[0, 2]$.

4. Scalar and vector partitions: Comparison

Hereafter, for comparative purposes, we denote the vector partition of the PS by $PS^*(\mathbf{r}, \mathbf{d})$, where $PS^*(\mathbf{r}, \mathbf{d})$

⁵ Eq. (1) is valid only if the set of S distinct scalar forecasts r^s ($s=1, \dots, S$) includes the values zero and one and if the difference between adjacent probability values is constant. For example, if this difference is 0.1 in a three-state ($N=3$) situation, then $S=11$ and $r^s = 0.0(0.1)1.0$ and, as a result, $T=66$ and $\mathbf{r}^t = (1.0, 0.0, 0.0), (0.9, 0.1, 0.0), \dots, (0.0, 0.0, 1.0)$.

$= (1/N)PS(\mathbf{r}, \mathbf{d})$. The range of $PS^*(\mathbf{r}, \mathbf{d})$ is the closed interval $[0, 2/N]$.

The scalar partition of the PS , $PS(r,d)$, in (2), can be expressed as

$$PS(r,d) = (1/M) \left[\sum_{s=1}^S M^s (r^s)^2 - 2 \sum_{s=1}^S M^s r^s \bar{d}^s + \sum_{s=1}^S M^s (\bar{d}^s)^2 \right] + (1/M) \left[\sum_{s=1}^S M^s \bar{d}^s - \sum_{s=1}^S M^s (\bar{d}^s)^2 \right]. \quad (4)$$

We denote the two sets of terms on the RHS of (4) by $S1$ and $S2$, respectively, and the terms which constitute these sets by $S11$, $S12$, and $S13$ and $S21$ and $S22$, respectively. Thus,

$$PS(r,d) = S1 + S2, \quad (5)$$

where

$$S1 = S11 + S12 + S13, \quad (6)$$

and

$$S2 = S21 + S22. \quad (7)$$

Note that $S13 = -S22$. Thus, $PS(r,d)$, in (5), can be expressed as

$$PS(r,d) = S11 + S12 + S21. \quad (8)$$

The vector partition of the PS , $PS^*(\mathbf{r}, \mathbf{d})$ [see Eq. (3)], can be expressed as

$$PS^*(\mathbf{r}, \mathbf{d}) = (1/NK) \times \left[\sum_{t=1}^T K^t \mathbf{r}^t (\mathbf{r}^t)' - 2 \sum_{t=1}^T K^t \mathbf{r}^t (\bar{\mathbf{d}}^t)' + \sum_{t=1}^T K^t \bar{\mathbf{d}}^t (\bar{\mathbf{d}}^t)' \right] + (1/NK) \left[\sum_{t=1}^T K^t \bar{\mathbf{d}}^t \mathbf{u}' - \sum_{t=1}^T K^t \bar{\mathbf{d}}^t (\bar{\mathbf{d}}^t)' \right],$$

or

$$PS^*(\mathbf{r}, \mathbf{d}) = (1/NK) \left[\sum_{t=1}^T K^t \sum_{n=1}^N (r_n^t)^2 - 2 \sum_{t=1}^T K^t \sum_{n=1}^N r_n^t \bar{d}_n^t + \sum_{t=1}^T K^t \sum_{n=1}^N (\bar{d}_n^t)^2 \right] + (1/NK) \left[\sum_{t=1}^T K^t \sum_{n=1}^N \bar{d}_n^t - \sum_{t=1}^T K^t \sum_{n=1}^N (\bar{d}_n^t)^2 \right]. \quad (9)$$

We denote the two sets of terms on the RHS of (9) by $V1$ and $V2$, respectively, and the terms which constitute these sets by $V11$, $V12$, and $V13$ and $V21$ and $V22$, respectively. Thus,

$$PS^*(\mathbf{r}, \mathbf{d}) = V1 + V2, \quad (10)$$

where

$$V1 = V11 + V12 + V13, \quad (11)$$

and

$$V2 = V21 + V22. \quad (12)$$

Note that $V13 = -V22$. Thus, $PS^*(\mathbf{r}, \mathbf{d})$, in (10), can

TABLE 1. A sample collection of forecasts and the relevant observations for a three-state ($N=3$) variable when the forecasts and observations are considered to be (a) scalars and (b) vectors.

a. Scalar forecasts and observations		
Forecast/ observation number m	Forecast r_m	Observation d_m
1	0.1	0
2	0.3	0
3	0.6	1
4	0.1	0
5	0.7	1
6	0.2	0
7	0.3	0
8	0.5	1
9	0.2	0
10	0.5	0
11	0.4	1
12	0.1	0
13	0.7	1
14	0.3	0
15	0.0	0
16	0.6	0
17	0.1	0
18	0.3	1
19	0.5	1
20	0.4	0
21	0.1	0
22	0.1	0
23	0.8	1
24	0.1	0
25	0.1	0
26	0.6	0
27	0.3	1
28	0.1	0
29	0.7	0
30	0.2	1

b. Vector forecasts and observations		
Forecast/ observation number k	Forecast r_k	Observation d_k
1	(0.1,0.3,0.6)	(0,0,1)
2	(0.1,0.7,0.2)	(0,1,0)
3	(0.3,0.5,0.2)	(0,1,0)
4	(0.5,0.4,0.1)	(0,1,0)
5	(0.7,0.3,0.0)	(1,0,0)
6	(0.6,0.1,0.3)	(0,0,1)
7	(0.5,0.4,0.1)	(1,0,0)
8	(0.1,0.8,0.1)	(0,1,0)
9	(0.1,0.6,0.3)	(0,0,1)
10	(0.1,0.7,0.2)	(0,0,1)

be expressed as

$$PS^*(\mathbf{r},\mathbf{d}) = V11 + V12 + V21. \tag{13}$$

We compare the expressions for the scalar partition $PS(r,d)$, in (4), and the vector partition $PS^*(\mathbf{r},\mathbf{d})$, in (9), term by term in the Appendix. We show that

$S11 = V11$, $S12 = V12$, and $S21 = V21$. Thus, from (8) and (13),

$$PS(r,d) = PS^*(\mathbf{r},\mathbf{d}). \tag{14}$$

We also show that the difference between $S13$ and $V13$, i.e., $V13 - S13$, is

$$V13 - S13 = (1/M) \left[\sum_{s=1}^S (1/M^s) \sum_{t=1}^{T^s} (K^{t,s})^2 \times \sum_{n=1}^{N^{t,s}-1} \sum_{n'=n+1}^{N^{t,s}} (\bar{d}_n^{t,s} - \bar{d}_{n'}^{t,s})^2 + \sum_{s=1}^S (1/M^s) \times \sum_{t=1}^{T^s-1} \sum_{t'=t+1}^{T^s} K^{t,s} K^{t',s} \sum_{n=1}^{N^{t,s}} \sum_{n'=1}^{N^{t',s}} (\bar{d}_n^{t,s} - \bar{d}_{n'}^{t',s})^2 \right], \tag{15}$$

where T^s denotes the number of distinct forecasts \mathbf{r}^t in the collection of vector forecasts of concern for which $r_n^t = r^s$ for some n ($t=1, \dots, T^s$); $N^{t,s}$ denotes the number of states in \mathbf{r}^t for which $r_n^t = r^s$ ($n=1, \dots, N^{t,s}$); $K^{t,s}$ denotes the number of forecasts \mathbf{r}_k in the subcollection of K^t vector forecasts for which $\mathbf{r}_k = \mathbf{r}^t$ ($k=1, \dots, K^{t,s}$), in which $r_n^t = r^s$ for some n ; $\bar{d}_k^{t,s}$ denotes an arbitrary observation in the relevant subcollection of $K^{t,s}$ vector observations, where $\mathbf{d}_k^{t,s} = (d_{1k}^{t,s}, \dots, d_{Nk}^{t,s})$ ($k=1, \dots, K^{t,s}$); and

$$\bar{d}_n^{t,s} = (1/K^{t,s}) \sum_{k=1}^{K^{t,s}} d_{nk}^{t,s}.$$

Note, from (15), that

$$V13 - S13 \geq 0. \tag{16}$$

Further, note that equality holds, in (16), only if either $N^{t,s} = 1$ or $\bar{d}_n^{t,s} = \bar{d}_{n'}^{t,s}$ for all n, n', t , and s in $D1$, the first term on the RHS of (15), and if $\bar{d}_n^{t,s} = \bar{d}_{n'}^{t',s}$ for all n, n', t, t' , and s in $D2$, the second term on the RHS of (15). Finally, note, from (6) and (11), that

$$V1 - S1 = V13 - S13 \geq 0, \tag{17}$$

and from (7) and (12), since $S13 = -S22$ and $V13 = -V22$, that

$$V2 - S2 = -(V13 - S13) \leq 0. \tag{18}$$

Thus, as in the two-state ($N=2$) situation, the values of the reliability and resolution terms for scalar forecasts are, in general, less and greater, respectively, than the values of the reliability and resolution terms for vector forecasts in N -state ($N > 2$) situations. That is, if a collection of forecasts is considered to consist of scalar forecasts, then the collection will appear, in general, to have more reliability and less resolution than if the collection is considered to consist of vector forecasts.

5. Scalar and vector partitions: A sample collection of forecasts

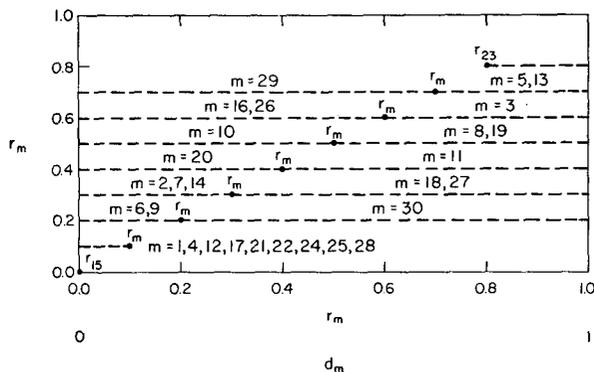
In order to illustrate the differences between the scalar and vector partitions of the PS in N -state ($N > 2$)

TABLE 2. The scalar partition of the PS for the sample collection of forecasts presented in Table 1a.

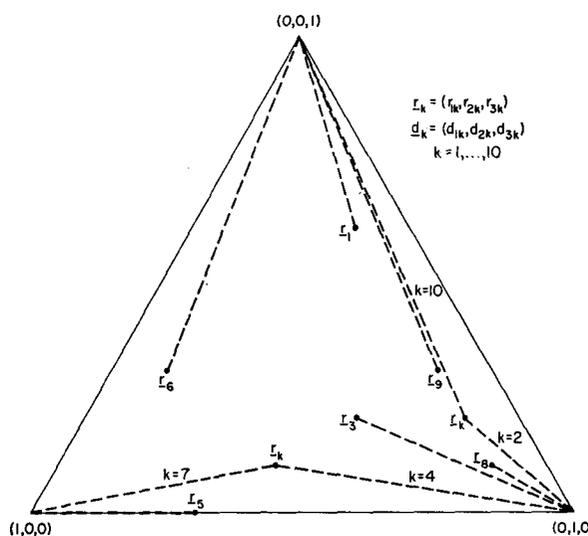
Subcollection number <i>s</i>	Forecast <i>r^s</i>	Number of forecasts <i>M^s</i>	Observed relative frequency <i>d^s</i>	Subcollection reliability <i>M^s(r^s - d^s)²</i>	Subcollection resolution <i>M^sd^s(1 - d^s)</i>
1	0.0	1	0.00	0.00	0.00
2	0.1	9	0.00	0.09	0.00
3	0.2	3	0.33	0.05(3)	0.66(7)
4	0.3	5	0.40	0.05	1.20
5	0.4	2	0.50	0.02	0.50
6	0.5	3	0.67	0.08(3)	0.66(7)
7	0.6	3	0.33	0.21(3)	0.66(7)
8	0.7	3	0.67	0.00(3)	0.66(7)
9	0.8	1	1.00	0.04	0.00
Total		30		0.55(3)	4.36(7)
Average				0.018(4)	0.145(6)

situations, we consider a sample collection of probability forecasts for a three-state ($N=3$) variable. The forecasts and the relevant observations are presented in Tables 1a and 1b, in which we identify these quantities as scalars and vectors, respectively. We depict the forecasts as scalars and vectors within the appropriate frameworks in Figs. 2a and 2b, respectively.

The scalar partition for these forecasts is presented in Table 2. Note that the values of the terms $S1$ (reliability) and $S2$ (resolution) are 0.018(4) and 0.145(6), respectively, and that their sum, i.e., $PS(r,d)$, equals 0.164. The vector partition for these forecasts is presented in Table 3. Note that the values of the terms $V1$ (reliability) and $V2$ (resolution) are 0.097(3) and 0.066(7), respectively, and that their sum, i.e., $PS^*(r,d)$, also equals 0.164. Thus, as indicated in (17) and (18), $V1(0.0973) \geq S1(0.0184)$ and $V2(0.0667) \leq S2(0.1456)$. The difference between the terms $S1$ and $V1$, or equivalently $S2$ and $V2$, can be computed directly by means of (15), since $V1 - S1 = S2 - V2 = V13 - S13$ [see (17) and (18)]. The computation of this difference, i.e., $V13 - S13$, for the sample collection of forecasts is presented in Table 4.⁶ Note that $V13 - S13 = 0.078(9)$ (cf. Tables 2 and 3).



a.



b.

FIG. 2. The forecasts presented in Table 1 depicted within the appropriate framework when forecasts are considered to be (a) scalars and (b) vectors. The dashed lines represent the distances between the forecasts and the relevant observations in the respective frameworks. The PS for each forecast equals the square of this distance.

TABLE 3. The vector partition of the PS for the sample collection of forecasts presented in Table 1b.

Subcollection number <i>t</i>	Forecast <i>r^t</i>	Number of forecasts <i>K^t</i>	Observed relative frequency <i>d^t</i>	Subcollection reliability <i>K^t(r^t - d^t)(r^t - d^t)'</i>	Subcollection resolution <i>K^td^t(u - d^t)'</i>
1	(0.1,0.3,0.6)	1	(0.0,0.0,1.0)	0.26	0.00
2	(0.1,0.6,0.3)	1	(0.0,0.0,1.0)	0.86	0.00
3	(0.1,0.7,0.2)	2	(0.0,0.5,0.5)	0.28	1.00
4	(0.1,0.8,0.1)	1	(0.0,1.0,0.0)	0.06	0.00
5	(0.3,0.5,0.2)	1	(0.0,1.0,0.0)	0.38	0.00
6	(0.5,0.4,0.1)	2	(0.5,0.5,0.0)	0.04	1.00
7	(0.6,0.1,0.3)	1	(0.0,0.0,1.0)	0.86	0.00
8	(0.7,0.3,0.0)	1	(1.0,0.0,0.0)	0.18	0.00
Total		10		2.92	2.00
Average*				0.097(3)	0.066(7)

* This average is computed on the basis of $NK=30$ forecasts [see Eq. (9)].

⁶ For example, for $s=6$, $D1(6)=0.00$ (since $N^{t,s}=1$ for $t=1$ and 2) and $D2(6) = (\frac{1}{3})(1)(2)(1.0-0.5)^2 = 0.16(7)$.

TABLE 4. The difference between the terms S_{13} and V_{13} in the scalar and vector partitions of the PS , respectively, for the forecasts presented in Table 1 [see Eq. (15)].

s	r^s	M^s	T^s	$K^{t,s}$	$N^{t,s}$	$\bar{d}_n^{t,s}$	$D1(s)$	$D2(s)$
1	0.0	1	1	1	1	0.0	0.00	0.00
2	0.1	9	6	1,1,2, 1,2,1	1,1,1, 2,1,1	0.0,0.0,0.0,0.0, 0.0,0.0,0.0	0.00	0.00
3	0.2	3	2	2,1	1,1	0.5,0.0	0.00	0.16(7)
4	0.3	5	5	1,1,1, 1,1	1,1,1, 1,1	0.0,1.0,0.0, 1.0,0.0	0.00	1.20
5	0.4	2	1	1	1	0.5	0.00	0.00
6	0.5	3	2	1,2	1,1	1.0,0.5	0.00	0.16(7)
7	0.6	3	3	1,1,1	1,1,1	1.0,0.0,0.0	0.00	0.66(7)
8	0.7	3	2	2,1	1,1	0.5,1.0	0.00	0.16(7)
9	0.8	1	1	1	1	1.0	0.00	0.00
Total		30					0.00	2.36(7)
Average							0.00	0.078(9)

Note that for this sample collection of forecasts the resolution term is eight times as large as the reliability term according to the scalar partition, while the resolution term is only two-thirds as large as the reliability term according to the vector partition.⁷ Thus, while the differences between the terms in these partitions can be expected, in general, to decrease as the number of forecasts in a collection increases, substantial differences may occur, at least for small collections.

6. The use of scalar and vector partitions: Discussion

A number of questions arise in connection with the use of the scalar and vector partitions of the PS in N -state ($N > 2$) situations. For example:

- 1) Which partitions have been used by meteorologists in previous forecast evaluation studies?
- 2) How sensitive are the results of these studies to the particular partition used?
- 3) Which partition should an evaluator use in such studies?
- 4) What are the effects upon sample size when forecasts are considered to be vectors rather than scalars?

Several of these questions have been discussed in some detail for the two-state ($N=2$) situation in Part I (pp. 278-280).

With regard to the first question, we are aware of only one study, that by Sanders (1958, 1963), in which partitions of the PS have been applied to forecasts for N -state ($N > 2$) variables. In his study, which involved forecasts for many different variables, some of which were two-state variables and some of which were N -state variables, Sanders applied a special scalar, or vector, partition⁸ to the two-state forecasts and a scalar partition to the N -state forecasts.

⁷ In N -state ($N > 2$) as well as in two-state ($N=2$) situations the scalar and vector partitions are concerned with the reliability and resolution of individual probabilities and sets of probabilities, or forecasts, respectively (see Footnote 2).

⁸ The special scalar partition, which is based upon the consideration of the probabilities assigned to only one of the two states, is equivalent to the vector partition in the two-state situation (see Part I, pp. 278-279).

With regard to the second question, we have not, as yet, applied the scalar and vector partitions to any large collection of forecasts for N -state ($N > 2$) variables

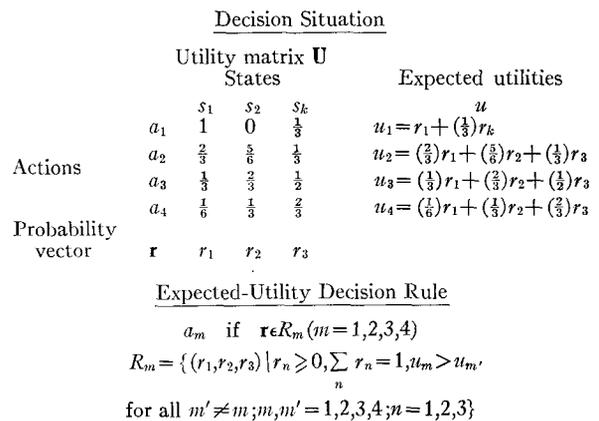


FIG. 3a. A four-action, three-state decision situation.

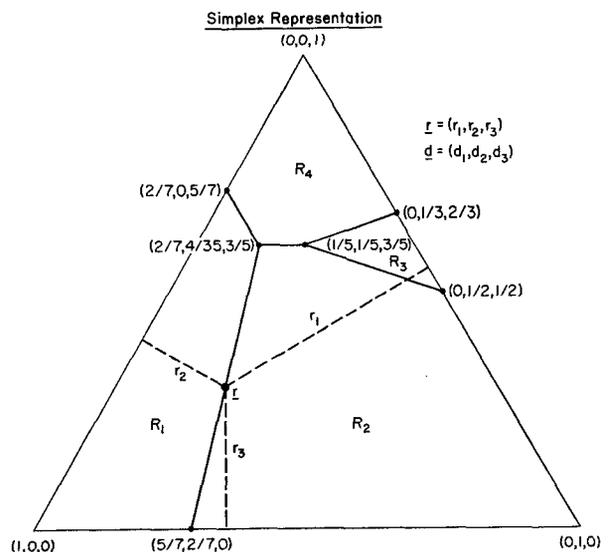


FIG. 3b. The decision situation in Fig. 3a depicted within the framework of a regular ($N-1$)-dimensional simplex, an equilateral triangle in a three-state ($N=3$) situation.

in order to determine the differences between the respective measures of reliability and resolution. However, as indicated in Section 5, these differences can be substantial, at least for small collections of forecasts.

With regard to the relative merits of these partitions, we indicated in Part I (pp. 279–280) that the vector partition appears to be more appropriate, in general, than the scalar partition in the two-state ($N=2$) situation from both a scientific and an economic point of view. We believe that these arguments are equally valid in N -state ($N>2$) situations. From a *scientific* point of view, evaluators are, or should be, primarily concerned with the reliability and resolution of forecasts rather than probabilities (see Footnote 7). In addition, the vector partition is, and the scalar partition is not, formulated in such a way that a one-to-one correspondence exists between the probabilities and the states for each subcollection of forecasts (see Part I, p. 279).

From an *economic* point of view, evaluators are, or should be, concerned with the reliability and resolution of forecasts in the vicinity of a decision maker's indifference hyperplanes (the sets of points, or forecasts, for which the decision maker is indifferent between two actions; see Murphy, 1972a). Since the regular ($N-1$)-dimensional simplex represents the proper framework within which to depict these hyperplanes (see Section 2; see also Murphy, 1972a), only the vector partition can provide the evaluator, or the decision maker, with the appropriate information. For example, consider the four-action, three-state decision situation described in Fig. 3a. This decision situation is depicted within the proper framework, an equilateral triangle in a three-state ($N=3$) situation, in Fig. 3b, in which the decision maker's indifference hyperplanes are line segments. Note, for example, that the decision maker is indifferent between actions a_1 and a_2 when the probabilities assigned to the states s_1 , s_2 , and s_3 are 0.5, 0.2, and 0.3, respectively [we denote this set of probabilities by the point $\mathbf{r}=(r_1, r_2, r_3)$ in Fig. 3b]. Thus, the reliability and resolution of the forecasts may be of particular concern when these, or similar, probabilities are assigned to the states. However, note that the evaluator, and the decision maker, will be concerned with the reliability and resolution of the *set* of probabilities, i.e., of the distinct vector forecast $\mathbf{r}'=(0.5, 0.2, 0.3)$, rather than with the reliability and resolution of the *individual* probabilities, i.e., of the distinct scalar forecasts $r^s=0.2$, $r^s=0.3$, and $r^s=0.5$.

With regard to the fourth question, consider a situation in which we have a collection of K vector forecasts for a three-state ($N=3$) variable and suppose that $K=100$ and $r^s=0.0(0.1)1.0$. Then, $M (=NK)=300$ and S and T , the number of distinct scalar and vector forecasts, equal 11 and 66 [see Eq. (1)], respectively. Thus, in the scalar framework we have 300 forecasts in 11 subcollections, while in the vector framework we have 100 forecasts in 66 subcollections. Therefore, for

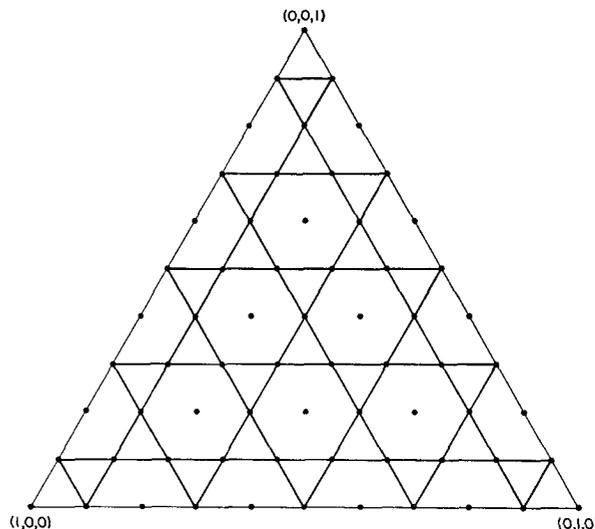


FIG. 4. A "procedure" for combining vector forecasts in a three-state ($N=3$) situation when the set of $S (=11)$ distinct scalar forecasts is $r^s=0.0(0.1)1.0$. For a complete description of this procedure, refer to the text.

small collections of forecasts the number of vector forecasts may not be sufficient to obtain reasonable estimates of the reliability and resolution of certain forecasts.

¶ One possible solution to this problem would be to combine those subcollections which correspond to "adjacent" forecasts with the subcollection which corresponds to the forecast of concern. Consider a three-state ($N=3$) situation in which $r^s=0.0(0.1)1.0$. Within the appropriate framework in this situation, an equilateral triangle, the first "tier" of six adjacent forecasts form a hexagon, in general, at the center of which is the forecast of concern. We depict this "procedure" in Fig. 4 for a subset of the set of 66 distinct vector forecasts; specifically, for the subset of 21 distinct vector forecasts obtained when $r^s=0.0(0.2)1.0$ [see Eq. (1)]. Note that the estimates of reliability and resolution for each forecast will be based upon 1) the six adjacent forecasts which form a hexagon for the interior points [(0.6, 0.2, 0.2), (0.4, 0.4, 0.2), etc.], 2) the four adjacent forecasts which form a trapezoid for the boundary points (excluding the vertices) [(0.8, 0.2, 0.0), (0.6, 0.4, 0.0), etc.], and 3) the two adjacent forecasts which, together with a vertex, form a triangle for the vertices [(1.0, 0.0, 0.0), (0.0, 1.0, 0.0), and (0.0, 0.0, 1.0)].⁹ This, or a similar, procedure would presumably provide reasonable estimates of these attributes for most, if not all, vector forecasts.¹⁰

⁹ The estimates of reliability and resolution for the other 45 distinct vector forecasts, 30 of which are interior points and 15 of which are boundary points, would be based upon similar combinations of forecasts.

¹⁰ This particular procedure may not provide reasonable estimates of reliability and resolution for certain forecasts; for example, for the boundary points and the vertices. In such a situation the second tier of six adjacent forecasts, which together with the first tier of forecasts form a star, in general, can also be combined with the forecast of concern.

Of course, many different procedures for combining forecasts can be formulated. For example, if an evaluator is concerned with the reliability and resolution of forecasts within the context of a particular decision situation, then the subcollections of vector forecasts should be combined in such a way that the estimates of these attributes for each forecast take account of the location of the decision maker's indifference hyperplanes.

7. Conclusion

In this paper we have described and compared scalar and vector partitions of the PS in N -state ($N > 2$) situations (see Footnote 3). These partitions, which are based upon expressions for the PS in which probability forecasts are considered to be scalars and vectors, respectively, provide similar, but not equivalent (i.e., linearly related), measures of the reliability and resolution of the forecasts. Specifically, the vector partition, when compared to the scalar partition, decreases the reliability and increases the resolution of the forecasts. A sample collection of forecasts has been used to illustrate the differences between these partitions.

We have briefly considered several questions related to the use of these partitions in N -state situations. Specifically, we have indicated that the vector partition is, in general, more appropriate than the scalar partition from both a scientific and an economic point of view and that the use of the vector, rather than the scalar, partition, in general, reduces the sample size of a collection of forecasts. In this regard, we have described a procedure for combining forecasts which should provide reasonable estimates of reliability and resolution for most, if not all, vector forecasts.

APPENDIX

Comparison of Terms in Scalar and Vector Partitions

We compare the scalar and vector partitions of the PS , $PS(r, \bar{d})$, in (4), and $PS^*(r, \mathbf{d})$, in (9), respectively, term by term in this Appendix. Since $S13 = -S22$ and $V13 = -V22$, we compare only $S11$ and $V11$, $S12$ and $V12$, $S13$ and $V13$, and $S21$ and $V21$. Note that, since $M = NK$, the coefficients of the respective terms are equal, and, as a result, need not be considered.

a. $S11$ and $V11$

Note that

$$S11 = \sum_{s=1}^S M^s (r^s)^2,$$

or

$$S11 = \sum_{m=1}^M r_m^2.$$

Further, note that

$$V11 = \sum_{t=1}^T K^t \sum_{n=1}^N (r_n^t)^2,$$

or

$$V11 = \sum_{k=1}^K \sum_{n=1}^N r_{nk}^2.$$

Then, since

$$\sum_{m=1}^M r_m^2 = \sum_{k=1}^K \sum_{n=1}^N r_{nk}^2,$$

$$S11 = V11.$$

b. $S12$ and $V12$

Note that

$$S12 = \sum_{s=1}^S M^s r^s \bar{d}^s,$$

or, since

$$\bar{d}^s = (1/M^s) \sum_{m=1}^{M^s} d_m^s,$$

$$S12 = \sum_{s=1}^S r^s \sum_{m=1}^{M^s} d_m^s,$$

or

$$S12 = \sum_{m=1}^M r_m d_m.$$

Further, note that

$$V12 = \sum_{t=1}^T K^t \sum_{n=1}^N r_n^t \bar{d}_n^t,$$

or, since

$$\bar{d}_n^t = (1/K^t) \sum_{k=1}^{K^t} d_{nk}^t,$$

$$V12 = \sum_{t=1}^T \sum_{n=1}^N r_n^t \sum_{k=1}^{K^t} d_{nk}^t,$$

or

$$V12 = \sum_{k=1}^K \sum_{n=1}^N r_{nk} d_{nk}.$$

Then, since

$$\sum_{m=1}^M r_m d_m = \sum_{k=1}^K \sum_{n=1}^N r_{nk} d_{nk},$$

$$S12 = V12.$$

c. S13 and V13

Note that

$$S13 = \sum_{s=1}^S M^s (\bar{d}^s)^2, \tag{A1}$$

and that

$$V13 = \sum_{t=1}^T K^t \sum_{n=1}^N (\bar{d}_n^t)^2. \tag{A2}$$

Let T^s denote the number of distinct forecasts r^t in the collection of vector forecasts of concern for which $r_n^t = r^s$ for some n ($t=1, \dots, T^s$); let $N^{t,s}$ denote the number of states in r^t for which $r_n^t = r^s$ ($n=1, \dots, N^{t,s}$); let $K^{t,s}$ denote the number of forecasts r_k in the subcollection of K^t vector forecasts for which $r_k = r^t$ ($k=1, \dots, K^{t,s}$), in which $r_n^t = r^s$ for some n ; and let $\mathbf{d}_k^{t,s}$ denote an arbitrary observation in the relevant subcollection of $K^{t,s}$ vector observations, where $\mathbf{d}_k^{t,s} = (d_{1k}^{t,s}, \dots, d_{Nk}^{t,s})$ ($k=1, \dots, K^{t,s}$). Note that

$$M^s = \sum_{t=1}^{T^s} N^{t,s} K^{t,s},$$

and that

$$\bar{d}_n^{t,s} = (1/K^{t,s}) \sum_{k=1}^{K^{t,s}} d_{nk}^{t,s}.$$

Then, S13, in (A1), can be expressed as

$$S13 = \sum_{s=1}^S \left(\sum_{t=1}^{T^s} N^{t,s} K^{t,s} \right) \times \left[\left(\sum_{t=1}^{T^s} \sum_{n=1}^{N^{t,s}} \sum_{k=1}^{K^{t,s}} d_{nk}^{t,s} \right) / \left(\sum_{t=1}^{T^s} N^{t,s} K^{t,s} \right) \right]^2,$$

or

$$S13 = \sum_{s=1}^S \left[1 / \left(\sum_{t=1}^{T^s} N^{t,s} K^{t,s} \right) \right] \left[\sum_{t=1}^{T^s} \sum_{n=1}^{N^{t,s}} \left(\sum_{k=1}^{K^{t,s}} d_{nk}^{t,s} \right)^2 + 2 \sum_{t=1}^{T^s} \sum_{n=1}^{N^{t,s}-1} \sum_{n'=n+1}^{N^{t,s}} \left(\sum_{k=1}^{K^{t,s}} d_{nk}^{t,s} \right) \left(\sum_{k=1}^{K^{t,s}} d_{n'k}^{t,s} \right) + 2 \sum_{t=1}^{T^s-1} \sum_{t'=t+1}^{T^s} \sum_{n=1}^{N^{t,s}} \sum_{n'=1}^{N^{t',s}} \left(\sum_{k=1}^{K^{t,s}} d_{nk}^{t,s} \right) \left(\sum_{k=1}^{K^{t',s}} d_{n'k}^{t',s} \right) \right],$$

or

$$S13 = \sum_{s=1}^S \left[1 / \left(\sum_{t=1}^{T^s} N^{t,s} K^{t,s} \right) \right] \left[\sum_{t=1}^{T^s} (K^{t,s})^2 \sum_{n=1}^{N^{t,s}} (\bar{d}_n^{t,s})^2 + 2 \sum_{t=1}^{T^s} (K^{t,s})^2 \sum_{n=1}^{N^{t,s}-1} \sum_{n'=n+1}^{N^{t,s}} (\bar{d}_n^{t,s}) (\bar{d}_{n'}^{t,s}) + 2 \sum_{t=1}^{T^s-1} \sum_{t'=t+1}^{T^s} K^{t,s} K^{t',s} \sum_{n=1}^{N^{t,s}} \sum_{n'=1}^{N^{t',s}} (\bar{d}_n^{t,s}) (\bar{d}_{n'}^{t',s}) \right]. \tag{A3}$$

Further, V13, in (A2), can be expressed as

$$V13 = \sum_{s=1}^S \sum_{t=1}^{T^s} K^{t,s} \sum_{n=1}^{N^{t,s}} \left[\left(\sum_{k=1}^{K^{t,s}} d_{nk}^{t,s} \right) / K^{t,s} \right]^2,$$

or

$$V13 = \sum_{s=1}^S \sum_{t=1}^{T^s} (1/K^{t,s}) \sum_{n=1}^{N^{t,s}} \left(\sum_{k=1}^{K^{t,s}} d_{nk}^{t,s} \right)^2,$$

or

$$V13 = \sum_{s=1}^S \left[1 / \left(\prod_{t=1}^{T^s} K^{t,s} \right) \right] \sum_{t=1}^{T^s} \left(\prod_{t'=1}^{T^s} K^{t',s} \right) \sum_{n=1}^{N^{t,s}} \left(\sum_{k=1}^{K^{t,s}} d_{nk}^{t,s} \right)^2,$$

or

$$V13 = \sum_{s=1}^S \sum_{t=1}^{T^s} K^{t,s} \sum_{n=1}^{N^{t,s}} (\bar{d}_n^{t,s})^2. \tag{A4}$$

The difference between S13, in (A3), and V13, in (A4), i.e., $V13 - S13$, is then

$$V13 - S13 = \sum_{s=1}^S \left[1 / \left(\sum_{t=1}^{T^s} N^{t,s} K^{t,s} \right) \right] \left\{ \left(\sum_{t=1}^{T^s} N^{t,s} K^{t,s} \right) \times \left[\sum_{t=1}^{T^s} K^{t,s} \sum_{n=1}^{N^{t,s}} (\bar{d}_n^{t,s})^2 \right] - \sum_{t=1}^{T^s} (K^{t,s})^2 \sum_{n=1}^{N^{t,s}} (\bar{d}_n^{t,s})^2 - 2 \sum_{t=1}^{T^s} (K^{t,s})^2 \sum_{n=1}^{N^{t,s}-1} \sum_{n'=n+1}^{N^{t,s}} (\bar{d}_n^{t,s}) (\bar{d}_{n'}^{t,s}) - 2 \sum_{t=1}^{T^s-1} \sum_{t'=t+1}^{T^s} K^{t,s} K^{t',s} \sum_{n=1}^{N^{t,s}} \sum_{n'=1}^{N^{t',s}} (\bar{d}_n^{t,s}) (\bar{d}_{n'}^{t',s}) \right\},$$

or

$$V13 - S13 = \sum_{s=1}^S \left[1 / \left(\sum_{t=1}^{T^s} N^{t,s} K^{t,s} \right) \right] \times \left\{ \sum_{t=1}^{T^s} K^{t,s} [(N^{t,s} - 1) K^{t,s} + \sum_{t'=1}^{T^s} N^{t',s} K^{t',s}] \times \sum_{n=1}^{N^{t,s}} (\bar{d}_n^{t,s})^2 - 2 \sum_{t=1}^{T^s} (K^{t,s})^2 \sum_{n=1}^{N^{t,s}-1} \sum_{n'=n+1}^{N^{t,s}} (\bar{d}_n^{t,s}) (\bar{d}_{n'}^{t,s}) - 2 \sum_{t=1}^{T^s-1} \sum_{t'=t+1}^{T^s} K^{t,s} K^{t',s} \sum_{n=1}^{N^{t,s}} \sum_{n'=1}^{N^{t',s}} (\bar{d}_n^{t,s}) (\bar{d}_{n'}^{t',s}) \right\},$$

or

$$V13 - S13 = \sum_{s=1}^S (1/M^s) \times \left[\sum_{t=1}^{T^s} (K^{t,s})^2 \sum_{n=1}^{N^{t,s}-1} \sum_{n'=n+1}^{N^{t,s}} (\bar{d}_n^{t,s} - \bar{d}_{n'}^{t,s})^2 + \sum_{t=1}^{T^s-1} \sum_{t'=t+1}^{T^s} K^{t,s} K^{t',s} \sum_{n=1}^{N^{t,s}} \sum_{n'=1}^{N^{t',s}} (\bar{d}_n^{t,s} - \bar{d}_{n'}^{t',s})^2 \right].$$

d. S21 and V21

Note that

$$S21 = \sum_{s=1}^S M^s \bar{d}^s,$$

or, since

$$\bar{d}^s = (1/M^s) \sum_{m=1}^{M^s} d_m^s,$$

$$S21 = \sum_{s=1}^S \sum_{m=1}^{M^s} d_m^s,$$

or

$$S21 = \sum_{m=1}^M d_m,$$

or

$$S21 = M/N.$$

Further, note that

$$V21 = \sum_{t=1}^T K^t \sum_{n=1}^N \bar{d}_n^t,$$

or, since

$$\bar{d}_n^t = (1/K^t) \sum_{k=1}^{K^t} d_{nk}^t,$$

$$V21 = \sum_{t=1}^T \sum_{n=1}^N \sum_{k=1}^{K^t} d_{nk}^t,$$

or

$$V21 = \sum_{k=1}^K \sum_{n=1}^N d_{nk},$$

or

$$V21 = K,$$

or, since $M = NK$,

$$V21 = M/N.$$

Thus,

$$S21 = V21.$$

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